Intro. to Mathematical Analysis I, 2nd Ed.

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LECTURE 1: SET THEORY

A set is a collection of elements

We write $x \in S$ to mean x is an element of S

Def^n/ $\emptyset = \{\} =$ empty set = set containing zero elements // $\underbrace{\{x\} = \text{set containing one element } x}_{\text{also called a "singleton"}}$

$\mathbb{N} = \{1, 2, 3, \dots\} =$ natural #'s $= \mathbb{Z}_+$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} = \{\dots, -2, -1, 0, 1, 2, \dots\} =$ integers

$\mathbb{N}_n = \{1, 2, 3, \dots, n\} =$ natural #'s from 1 to n.

$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\} =$ rational #'s

$\mathbb{R} = (-\infty, \infty) = \{x \mid x \text{ a real \#}\} =$ real #'s

$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\} =$ complex #'s

$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\} =$ non-negative integers

$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\} =$ closed interval from a to b

$(a, b) = \{x \in \mathbb{R} \mid a < x < b\} =$ open interval from a to b

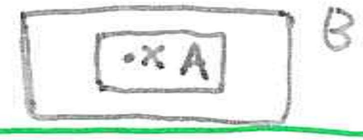
$[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\} =$ interval from a to infinity, a included

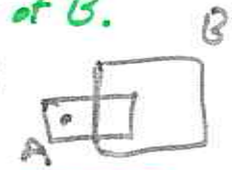
$(a, \infty) = \{x \in \mathbb{R} \mid x > a\} =$ interval from a to ∞, a excluded = open int. from a to ∞.

Can define $(-\infty, a]$ and $(-\infty, a)$ similarly, also $[a, b)$ and $(a, b]$.

Defn/ If A, B are sets then A is <u>subset</u> of B and we write $A \subseteq B$ iff $x \in A$ implies $x \in B$. (our text writes $A \subset B$ in place of $A \subseteq B$)

If $A \subseteq B$ and $A \neq B$ then we write $A \subsetneq B$ and say A is "proper" subset of B.





When two sets have the same elements then we say the sets are <u>equal</u>; $A = B$ iff $x \in A \Leftrightarrow x \in B$.

<u>Example: even and odd integers</u>

Defn/ $2\mathbb{Z} = \{2x \mid x \in \mathbb{Z}\}$ = set of even integers

$1 + 2\mathbb{Z} = \{2x+1 \mid x \in \mathbb{Z}\}$ = set of odd integers.

Both $2\mathbb{Z}$ and $1+2\mathbb{Z}$ are subsets of $\mathbb{Z}$.

<u>Proof</u>: If $z \in 2\mathbb{Z}$ then $z = 2x$ for some $x \in \mathbb{Z}$ and $2x \in \mathbb{Z}$ $\therefore$ $z = 2x \in \mathbb{Z}$ thus by definition of subset $2\mathbb{Z} \subseteq \mathbb{Z}$. //

If $z \in 1+2\mathbb{Z}$ then $\exists x \in \mathbb{Z}$ s.t. $z = 2x+1$. Observe $2x+1 \in \mathbb{Z}$ hence $z \in 1+2\mathbb{Z}$ implies $z \in \mathbb{Z}$ and it follows $1+2\mathbb{Z} \subseteq \mathbb{Z}$. //

<u>Question</u>: how to our examples fit together (or not)

$$\emptyset \subseteq \mathbb{N}_n \subseteq \mathbb{N} \subseteq \mathbb{Z}_{\geq 0} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

$(0,1) \subseteq [0,1) \subseteq [0,1] \subseteq [0,\infty) \subseteq \mathbb{R}$, however $[0,1) \nsubseteq (0,1]$
$(0,1] \nsubseteq [0,1)$

$\boxed{\text{Th}^{\underline{m}}/ \text{ For sets } A, B, \; A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A}$ ← Sets are equal if we can demonstrate each is a subset of the other; double-containment

Proof: Suppose $A = B$ then $x \in A \Leftrightarrow x \in B$

Thus $\underline{x \in A \Rightarrow x \in B}$ and $\underline{x \in B \Rightarrow x \in A}$ thus $\underline{A \subseteq B}$ and $\underline{B \subseteq A}$.

Conversely, suppose $A \subseteq B$ and $B \subseteq A$. If $x \in A$ then since $A \subseteq B$ we have $x \in B$.

Likewise, if $x \in B$ then since $B \subseteq A$ we have $x \in A$. Therefore, for each x we have $x \in A \Leftrightarrow x \in B$ and by def$^{\underline{n}}$ $A = B$.//

Comment: when proving something which seems obvious it is especially important to convince your audience (me) that you understand the definitions in use and how they fit into the logic of the proof. Sometimes, just writing the relevant definition alone is half the proof.

Def$^{\underline{n}}$/ If X is a set then $\mathcal{P}(X)$ is the set of all subsets of X. We call $\mathcal{P}(X)$ the POWER SET of X.

$\emptyset \subseteq X$ and $X \subseteq X$

Example: $X = \{1,2\}$ has $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$Y = \{a,b,c\}$ has $\mathcal{P}(Y) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

Can you see any pattern to the # of subsets in $\mathcal{P}(X)$?

SET OPERATIONS

Def^{n}/ Let A, B be sets then define

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} = \text{union of } A \,\&\, B$$

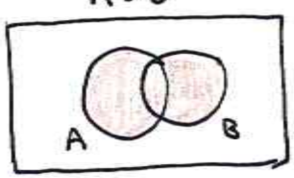
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} = \text{intersection of } A \,\&\, B$$

$$A - B = \{x \mid x \in A \text{ and } x \notin B\} = A \setminus B = \text{set difference}$$

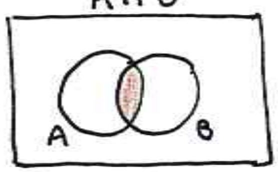
$$A \Delta B = (A - B) \cup (B - A) = \text{symmetric difference}$$

If X is the universal set and $A \subseteq X$ then $\underbrace{A^c = \overline{A} = X - A}_{\text{complement of } A \text{ in } X}$

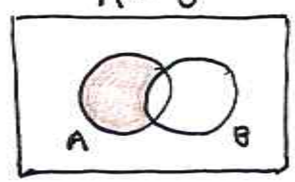
$A \cup B$



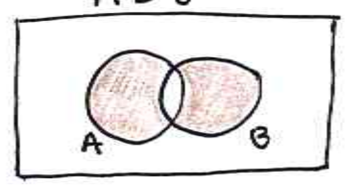
$A \cap B$



$A - B$



$A \Delta B$



CONJECTURE: $A \cup B = (A \Delta B) \cup (A \cap B)$ (I suspect this from looking at the pictures above)

$\boxed{\text{Def}^{n}/ \text{ If } A \cap B = \phi \text{ then } A \,\&\, B \text{ are disjoint sets}}$

PROPERTIES

For sets A, B, C which are subsets of the universal set X,

(a.) $A \cup \overline{A} = X$

(b.) $A \cap \overline{A} = \phi$

(c.) $\overline{\overline{A}} = A$

(h.) $A - B = A \cap \overline{B}$

d.) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

e.) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(d.–e.) distributive laws

f.) $A - (B \cup C) = (A - B) \cap (A - C)$

g.) $A - (B \cap C) = (A - B) \cup (A - C)$

(f.–g.) De Morgan's Laws

Remark: set $A = X$ then De Morgan's Laws: $\overline{B \cup C} = \overline{B} \cap \overline{C}$, $\overline{B \cap C} = \overline{B} \cup \overline{C}$

SELECT PROOFS OF SET OPERATION PROPERTIES

(b.) CLAIM: $A \cap \bar{A} = \emptyset$

~~Observe $\emptyset \subseteq A \cap \bar{A}$ is vacuously true since $\emptyset$ has no elements.~~ (no need)

Let $x \in A \cap \bar{A}$ then $x \in A$ and $x \in \bar{A}$. Thus x is in A and x is not in A. This is a contradiction hence $\nexists\, x \in A \cap \bar{A}$ which shows $A \cap \bar{A} = \emptyset$.

(d.) CLAIM: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Observe $x \in A \cap (B \cup C)$ means $x \in A$ and $x \in B \cup C$. But, also note that $x \in B \cup C$ means $x \in B$ or $x \in C$. Thus $x \in A$ and $x \in B$ OR $x \in A$ and $x \in C$. Hence $x \in A \cap B$ or $x \in A \cap C$. Consequently, $x \in (A \cap B) \cup (A \cap C)$. Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ *

Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$ by def^{n} of union. Thus $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$ by def^{n} of intersection. Hence $x \in A$ and $x \in B$ or $x \in C$ which means $x \in A$ and $x \in B \cup C$. Thus, $x \in A \cap (B \cup C)$ and we've shown $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ **

In conclusion, since we've shown double-containment with * and ** the claim $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Remark: perhaps the proof is better.

Proofs continued:

(d.) For every x,

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \text{ and } x \in B \cup C \\ &\iff x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\iff (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ &\iff x \in A \cap B \text{ or } x \in A \cap C \\ &\iff x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ as they have the same elements.

(h.) Claim: $A - B = A \cap \overline{B}$

Let $x \in A - B$ then $x \in A$ and $x \notin B$ by definition of $A - B$. Then $x \notin B$ implies $x \in \overline{B}$ by definition of complement of B. Thus, $x \in A$ and $x \in \overline{B} \Rightarrow x \in A \cap \overline{B}$. Therefore, $A - B \subseteq A \cap \overline{B}$.

Let $x \in A \cap \overline{B}$ then $x \in A$ and $x \in \overline{B}$. But, $x \in \overline{B}$ means $x \notin B$ so $x \in A$ and $x \notin B$ hence $x \in A - B$ and we've shown $A \cap \overline{B} \subseteq A - B$.

Concluding, $A - B \subseteq A \cap \overline{B}$ and $A \cap \overline{B} \subseteq A - B$ $\therefore$ $A - B = A \cap \overline{B}$.

SET OPERATIONS FOR FAMILIES OF SETS

Def^{n}/ A <u>family</u> <u>of</u> <u>sets</u> indexed by I is collection of sets A_i such that $i \in I$.

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for every } i \in I\}$$

we abbreviate to $\bigcup A_i$ and $\bigcap A_i$ when context is clear. Also for $I = \mathbb{N}$ we use notation $\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i \in \mathbb{N}} A_i = \bigcap_{i=1}^{\infty} A_i$

Examples:

(1.) $I = \mathbb{N}$, $A_j = [-j, j]$ gives $\bigcup_{j=1}^{\infty} [-j, j] = \mathbb{R}$ & $\bigcap_{j=1}^{\infty} [-j, j] = [-1, 1]$.

(2.) $J = (0, 1]$, $A_\lambda = (-\lambda, \lambda)$ gives $\underbrace{\bigcup_{\lambda \in J} (-\lambda, \lambda) = (-1, 1)}_{\text{infinite union of open intervals is open interval}}$ & $\underbrace{\bigcap_{\lambda \in J} (-\lambda, \lambda) = \{0\}}_{\text{infinite intersection of open intervals is closed set } \{0\}}$

(THESE COMMENTS FOR <u>FUTURE</u> USE AFTER WE TALK ABOUT TOPOLOGY LATER...)

PROPERTIES OF SET OPERATIONS ON FAMILIES OF SETS

Th^{m}/ Let $\{A_i \mid i \in I\}$ be indexed family of subsets of X and suppose $B \subseteq X$.

$$\left.\begin{aligned} &(a.)\quad B \cup \Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i \in I} B \cup A_i \\ &(b.)\quad B \cap \Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i \in I} B \cap A_i \end{aligned}\right\} \text{distributive properties}$$

$$\left.\begin{aligned} &(c.)\quad B - \bigcap A_i = \bigcup (B - A_i) \\ &(d.)\quad B - \bigcup A_i = \bigcap (B - A_i) \\ &(e.)\quad \overline{\bigcap A_i} = \bigcup \overline{A_i} \\ &(f.)\quad \overline{\bigcup A_i} = \bigcap \overline{A_i} \end{aligned}\right\} \text{De Morgan's Laws}$$

<u>Proof</u>: I'll prove (c.) to begin,

$$\begin{aligned} x \in B - \bigcap A_i &\Leftrightarrow x \in B \text{ and } x \notin \bigcap A_i \quad * \text{ (needs some explaining)} \\ &\Leftrightarrow x \in B \text{ and } x \notin A_i \text{ for some } i \in I \\ &\Leftrightarrow x \in (B - A_i) \text{ for some } i \in I \\ &\Leftrightarrow x \in \bigcup (B - A_i) \end{aligned}$$

$\therefore B - \bigcap A_i = \bigcup (B - A_i)$ as they have the same elements. //

* Notice $x \notin \bigcap A_i$ means $x \in A_i$ for all $i \in I$ is impossible. Thus there is at least one $i \in I$ for which $x \notin A_i$.

I'll leave the other parts for homework. Hint, (e.) and (f.) follow easily from (c.), (d.).

CARTESIAN PRODUCT OF SETS

We may construct ordered pairs $(a,b) = \{a, \{a,b\}\}$ if we wish. The net-effect is that for $(a,b) = (c,d)$ we need $a = c$ <u>and</u> $b = d$. Generally, $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$ iff $a_i = b_i \ \forall i = 1, 2, \dots, n$. We say $(a_1, a_2, \dots, a_n)$ is an "n-tuple". I'll leave the careful construction to a different course. We'll assume this paragraph makes sense.

Def^n/ $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$ = CARTESIAN PRODUCT OF A & B

$$\bigtimes_{i=1}^{n} A_i = A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \dots, x_n) \mid x_i \in A_i \ \forall i = 1, 2, \dots, n\}$$

when $\underbrace{A_1 = A_2 = \dots = A_n}_{\text{all } A}$ we write $\bigtimes_{i=1}^{n} A_i = A^n$

PROPERTIES: Assume A, B, C, D are sets,

(a.) $A \times (B \cup C) = \cancel{(A \cup B) \times (A \cup C)}$ Sorry ☺ $= (A \times B) \cup (A \times C)$

(b.) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(c.) $A \times \emptyset = \emptyset$

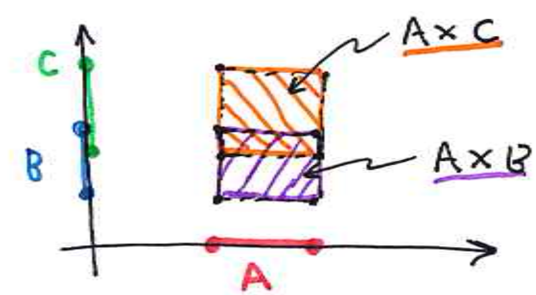
(d.) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(e.) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$

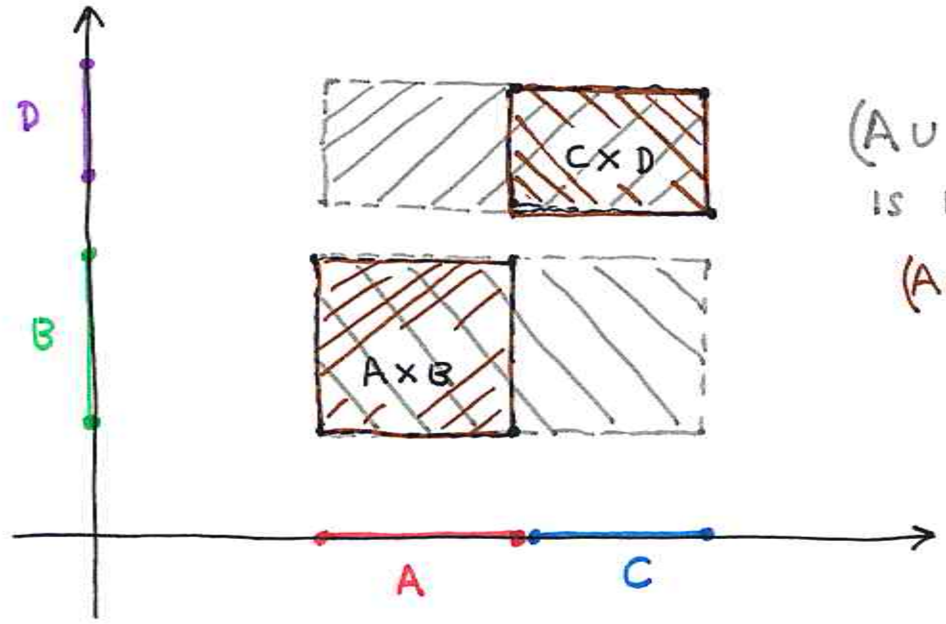
Remark: technically $A \times (A \times A) \neq (A \times A) \times A$

$(x, (y,z))$ vs. $((x,y), z)$

but, it is usually our custom to ignore these distinctions and simply write $(x,y,z) \in A \times A \times A$.



VISUALIZATION OF PROPERTY $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$



$(A \cup C) \times (B \cup D)$

IS LARGER THAN

$(A \times B) \cup (C \times D)$

Remark: $B \cap D = \emptyset$
and $(A \times B) \cap (C \times D) = \emptyset$
so the intersection property
vacuously illustrated here

VISUALIZATION OF $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$