LECTURE 20 : SEQUENCES & NETS & CONVERGENCE

Def

Def^{n}/ A topological space is <u>2^{nd} COUNTABLE</u> if the topology admits a countable basis of open sets

Def^{n}/ A topological space is called <u>separable</u> if it contains a countable dense subset

<u>Lemma 6.6</u>: Every 2^{nd} countable space is separable

<u>Lemma 6.7</u>: Any separable metric space has a countable basis

$\boxed{EO}$ $\ell^2(\mathbb{R})$ set of real sequences $\{a_n\}$ s.t. $\sum a_n^2 < \infty$ forms metric space via $d(a,b) = \|a-b\|$ where $\|a\| = \sqrt{\sum_{n=1}^{\infty} a_n^2}$ the set of eventually null, rational sequences are dense in $\ell^2(\mathbb{R})$ thus $\ell^2(\mathbb{R})$ is separable $\Rightarrow$ $\ell^2(\mathbb{R})$ has a countable basis. (topologically speaking)

<u>Proposition 6.9</u>: In a 2^{nd} countable space every open cover admits a countable subcover.

Def^{n}/ A topological space satisfies the <u>1^{st} Countability Axiom</u>, or is <u>1^{st} countable</u> if every point admits a countable local basis of nbhds.

<u>Lemma 6.11</u>: Metric spaces are first countable

Proof: $\{B(x, 2^{-n}) \mid n \in \mathbb{N}\}$ gives local basis of nbhds at x in metric space.

<u>Th^{m} 6.13</u>: X a 2^{nd} countable, locally compact Hausdorff space, $\exists$ an exhaustion of X by compact sets; that is, a sequence $K_1 \subset K_2 \subset \cdots$ of compact sets covering X such that $K_n \subseteq K_{n+1}^{\circ}$ for any $n \in \mathbb{N}$.

SEQUENCES

(§6.2 in Manetti)

Def^n/ A **sequence** in a topological space X is a map $a : \mathbb{N} \to X$ where we use notation $a(i) = a_i$ typically and also $\{a_n\}$ for $a : \mathbb{N} \to X$;

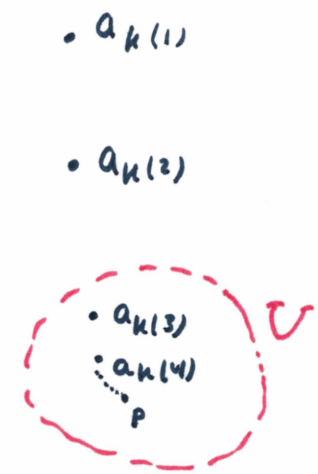
(1.) The sequence converges to $p \in X$ if for any nbhd U of p there is an $N \in \mathbb{N}$ s.t. $a_n \in U$ $\forall n \geq N$.

(2.) A point $p \in X$ is called a limit point of the sequence if for any nbhd U of p, and for every $N \in \mathbb{N}$, there exists $n \geq N$ such that $a_n \in U$. We say $\{a_n\}$ accumulates at p

Sequential limits are unique in Hausdorff spaces. Consider, sequence $\{a_n\}$ which converges to distinct $p \neq q$ in the Hausdorff space X. Since X Hausdorff, $\exists\, U, V$ open in X with $p \in U$, $q \in V$ and $U \cap V = \emptyset$. Since $a_n \to p$ and $a_n \to q$, $\exists\, N, M \in \mathbb{N}$ s.t. $a_n \in U$ $\forall n \geq N$ and $a_n \in V$ $\forall n \geq M$. Suppose $n \geq \max(N, M)$ then $a_n \in U \cap V \rightarrow\!\leftarrow U \cap V = \emptyset$.

Def^n/ A sequence $\{a_n\}$ in X is said to be convergent if it converges to some point $p \in X$. When X is Hausdorff, we say p is the limit of $\{a_n\}$ and write $a_n \to p$ or $\lim_{n \to \infty}(a_n) = p$.

Def^{n}/ A <u>subsequence</u> of a sequence $a : \mathbb{N} \to X$ is the composite of the mapping a with some strictly increasing function $k : \mathbb{N} \to \mathbb{N}$



<u>Lemma 6.17</u>: Let $\{a_n\}$ be a sequence in X. If $\exists$ subsequence $\{a_{k(n)}\}$ converging to $p \in X$, then p is a limit point for $\{a_n\}$

<u>Proof</u>: Let U be nbhd of p. By def^{n} of $\{a_{k(n)}\}$ converging to $p \in X$ $\exists\ N \in \mathbb{N}$ s.t. $a_{k(n)} \in U$ $\forall n \geq N$. Let $M \in \mathbb{N}$ and choose $m = \max\{k(N), M\} \Rightarrow a_m \in U$ where $m \geq M$ thus p is limit pt. //

<u>Prop. 6.18</u>: Let X be a 1st countable space, $A \subseteq X$. For every $x \in X$, TFAE,

(1.) $\exists$ a sequence with values in A that converges to x

(2.) the point x is a limit point of an A-valued sequence

(3.) the point x belongs in the closure of A

<u>Proof</u>: p. 110 Manetti. //

<u>Remark</u>: in some analysis courses you might see the closure of a set A <u>defined</u> by the set of all limit points of A. Here topology (open sets) are basic whereas sequential idea is 2ndary.

Lemma 6.19: In a compact space, any sequence has limit points

Proof: Let X be compact and $a : \mathbb{N} \to X$ a sequence. For every m we define $C_m = \overline{\{a_n \mid n \geq m\}} \subseteq X$. Then $x \in X$ is limit pt. for a if $x \in C_m$ for every m. Notice that C_m are compact and $C_1 \supseteq C_2 \supseteq C_3 \supseteq C_4 \supseteq \cdots$ is countable descending chain of non-empty closed sets $\Rightarrow \bigcap_{m=1}^{\infty} C_m \neq \emptyset$ by Prop. 4.46 (p. 75). //

Def^n/ A space X is sequentially compact if every sequence has a convergent subsequence.

Lemma 6.21: A 1st countable space is sequentially compact iff every sequence admits a limit point. In particular, any 1st countable compact space is also sequentially compact.

Proof: p. 110 Manetti.

Proposition 6.22: TFAE in a 2nd countable space X,

(1.) X is compact
(2.) every sequence in X has a limit pt.
(3.) X is sequentially compact

Remark: in general compact $\not\Rightarrow$ sequentially compact (Ex. 7.7)
sequentially compact $\not\Rightarrow$ compact (Ex. 7.8)

Cauchy Sequences (§6.3)

Def^{n}/ A sequence $\{a_n\}$ in a <u>metric space</u> (X, d) is a <u>CAUCHY SEQUENCE</u> if for any $\varepsilon > 0$ there exists an integer N such that $d(a_n, a_m) < \varepsilon$ $\forall n, m \geq N$.

For fixed m, $\lim_{n \to \infty} d(a_n, a_m) = 0$

Th^{m}/ If $a_n \to L$ as $n \to \infty$ then $\{a_n\}$ is Cauchy Seq.

Proof: Suppose $a_n \to L$. Then if $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_n \in B(L, \varepsilon/2)$ for all $n \geq N$. Then apply Δ-ineq,

$$d(a_n, a_m) \leq d(a_n, L) + d(L, a_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $n, m \geq N$. thus $\{a_n\}$ is Cauchy sequence. //

Lemma 6.25: A CAUCHY SEQUENCE IS CONVERGENT iff it has limit points. In particular, any Cauchy Sequence in a sequentially compact metric space converges.

Proof: suppose $\{a_n\}$ is Cauchy Seq. in metric space (X, d), and suppose $\{a_n\}$ has limit point $P \in X$. By Cauchy condition, given $\varepsilon > 0$ we may select $N \in \mathbb{N}$ for which $d(a_m, a_n) < \varepsilon/2$ for all $m, n \geq N$. Since P is limit pt. there is also $M \geq N$ for which $d(P, a_N) < \varepsilon/2$. Thus,

$$d(P, a_n) \leq d(P, a_N) + d(a_N, a_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $n \geq M$. Thus $a_n \to P$ as $n \to \infty$. //

Def$\underline{n}$/ A metric space is called <u>complete</u> if every Cauchy sequence converges to a point in X

Since Lemma 6.21 indicates every compact metric space is sequentially compact we find from Lemma 6.25 that every compact metric space is complete. Some very much <u>not</u> compact examples of complete metric spaces,

Th$\underline{m}$/ The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete metric spaces

<u>Proof</u>: Since $\mathbb{C}^n = \mathbb{R}^{2n}$ we may focus on $\mathbb{R}^n$. Let $\{a_n\}$ be Cauchy Seq. in $\mathbb{R}^n$ and choose $N \in \mathbb{N}$ for which

$$\|a_n - a_N\| < 1$$

for all $n \geq N$. Construct $R = \max\{\|a_1\|, \|a_2\|, \dots, \|a_N\|\}$ we observe $\|a_n - a_N\| < 1$ implies $\{a_n\}$ is subset of compact space

$$D = \{x \in \mathbb{R}^n \mid \|x\| \leq R+1\}$$

thus by Lemma 6.25 $\{a_n\}$ converges $\Rightarrow$ $\mathbb{R}^n$ complete. //

<u>Remark</u>: I give a low-tech proof of <u>this in Advanced Calculus</u> p. 113 which, in my view, is easier. // <u>Remark</u>: I skipped Lemma 6.28 and $[0,1]$ vs. $\mathbb{R}$

<u>Prop. 6.29</u>: A subspace in a complete metric space is closed $\Leftrightarrow$ the subspace is complete w.r.t. the induced metric.

Compact Metric Spaces (§6.4)

Def^{n}/ A metric space is said to be <u>totally bounded</u> if it can be covered by a finite # of open balls of radius r, for any positive real # r.

A totally bounded metric space is bounded (take max over finite # of balls)

<u>Lemma 6.31</u>: Every sequentially compact metric space is totally bounded.

<u>Lemma 6.32</u>: Every totally bounded metric space is 2^{nd} countable.

<u>Th^{m} 6.33</u>: In a metric space X, TFAE,

(1.) X is compact
(2.) every sequence in X has a limit point
(3.) X is sequentially compact
(4.) X is complete and totally bounded

Furthermore, if any one of the above holds, then X is 2^{nd} countable.

<u>Lemma 6.34</u>: A subspace A in a metric space is totally bounded $\Leftrightarrow$ $\bar{A}$ is totally bounded.

Def^{n}/ A subspace A of a top. space X is <u>relatively compact</u> if it is contained in a compact subspace of X.

<u>Corollary 6.36</u>: A subspace in a complete metric space is relatively compact iff it is totally bounded.

DIRECTED SETS AND NETS (GENERALIZED SEQUENCES)

§6.8 Manetti:

Remark: skipped §6.5, 6.6 and 6.7 which concern Baire spaces & completion.

NETS allow the concept of a sequence in the context of topological spaces which are not 1st countable,

Defn/ An <u>ordered</u> set $(I, \leq)$ is called <u>directed</u> when for every $i, j \in I$, there exists $h \in I$ such that $h \geq i$ and $h \geq j$. Equivalently, a directed set is an ordered set in which every finite subset is upper bounded.

$\boxed{E1}$ $\mathbb{N}$ with standard order

not the lexographic ordering I expected.

$\boxed{E2}$ $\mathbb{N}^n$ with $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \iff a_i \leq b_i \;\; \forall i = 1, 2, \ldots, n$

$\boxed{E3}$ $\mathcal{P}_0(A)$, the set of all finite subsets of A, ordered via <u>inclusion</u>

$\boxed{E4}$ $\mathcal{I}(x)$ the set of nbhds of a pt. x in topological space X with the order relation $U \leq V \iff V \subseteq U$.

Defn/ A map $p : J \to I$ between directed sets is called a <u>cofinal</u> <u>morphism</u> if it preserves orderings ($p(j) \geq p(j')$ if $j \geq j'$) and for every $i \in I$ there exists $j \in J$ such that $p(j) \geq i$.

Def^{n}/ A <u>NET</u> in a topological space $\mathfrak{X}$ is a map $f : I \longrightarrow \mathfrak{X}$ where I is directed.

ES/ a sequence is a net

EG/ $a : \mathbb{N}^n \longrightarrow \mathfrak{X}$ could denote as $a_{i_1 i_2 \cdots i_n}$, this is the origin of the term "net", maybe $n=2$ moreso, I think nets arise naturally in the theory of integration.

Def^{n}/ Let $\mathfrak{X}$ be topological space, $f : I \longrightarrow \mathfrak{X}$ a net and x a point in $\mathfrak{X}$. We say:

(1.) the net f <u>converges to x</u> if for any nbhd $U \in \mathcal{I}(x)$ there exists index $i \in I$ such that $f(j) \in U$ $\forall j \geq i$.

(2.) x is a <u>limit point of the net f</u> if for any nbhd $U \in \mathcal{I}(x)$ and any $i \in I$ there exists $j \geq i$ s.t. $f(j) \in U$.

A converging net must converge to a limit pt, but a limit pt. isn't necessarily a point to which the net converges.

<u>Lemma</u> 6.60: Let $f : I \longrightarrow \mathfrak{X}$ be a net. Then $x \in \mathfrak{X}$ is a limit pt. for f iff there exists a cofinal morphism $p : J \longrightarrow I$ such that the net $f \circ p : J \longrightarrow \mathfrak{X}$ converges to x.

<u>Proposition 6.61</u>: Let A be subset of topological space X. For any point $x \in X$, TFAE:

(1.) there is a net in A that converges to x,

(2.) there is a net in A with limit point x,

(3.) the point x belongs in the closure of A

<u>Th^m 6.62</u>: A topological space X is compact iff every net has a limit point

$\boxed{E7}$ A topological vector space V is a vector space with a Hausdorff topology where the vector addition and scalar multiplication are continuous. Manetti on p. 127 → 128 sketches argument via nets to prove <u>every finite-dim'l subspace of a topological vector space is closed</u>.

<u>Remark</u>: the above example might have application to my topological problem in supermanifolds...