Prove that the equation

$$\sum_{k=0}^{n} a_k \cos(2k+1)x = 0$$

has a solution on $(0, \frac{\pi}{2})$.

**4.2.7** Let $f\colon [0,\infty) \to \mathbb{R}$ be a differentiable function. Prove that if both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist, then $\lim_{x\to\infty} f'(x) = 0$

**4.2.8** $\triangleright$ Let $f\colon [0,\infty) \to \mathbb{R}$ be a differentiable function.

(a) Show that if $\lim_{x\to\infty} f'(x) = a$, then $\lim_{x\to\infty} \dfrac{f(x)}{x} = a$.

(b) Show that if $\lim_{x\to\infty} f'(x) = \infty$, then $\lim_{x\to\infty} \dfrac{f(x)}{x} = \infty$.

(c) Are the converses in part (a) and part (b) true?

LECTURE 21: APPLICATIONS OF MVT

## 4.3 SOME APPLICATIONS OF THE MEAN VALUE THEOREM

In this section, we assume that $a, b \in \mathbb{R}$ and $a < b$. In the proposition below, we show that it is possible to use the derivative to determine whether a function is constant. The proof is based on the Mean Value Theorem.

**Proposition 4.3.1** Let $f$ be continuous on $[a,b]$ and differentiable on $(a,b)$. If $f'(x) = 0$ for all $x \in (a,b)$, then $f$ is constant on $[a,b]$.

**Proof:** Suppose by contradiction that $f$ is not constant on $[a,b]$. Then there exist $a_1$ and $b_1$ such that $a \le a_1 < b_1 \le b$ and $f(a_1) \neq f(b_1)$. By Theorem 4.2.3, there exists $c \in (a_1, b_1)$ such that

$$f'(c) = \frac{f(b_1) - f(a_1)}{b_1 - a_1} \neq 0,$$

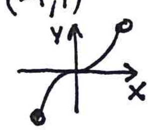
which is a contradiction. $\square$

The next application of the Mean Value Theorem concerns developing simple criteria for monotonicity of real-valued functions based on the derivative.

$f(x) = x^3$ on $(-1,1)$

$f'(x) = 3x^2$

$f'(0) = 0$




**Proposition 4.3.2** Let $f$ be differentiable on $(a,b)$.

(i) If $f'(x) > 0$ for all $x \in (a,b)$, then $f$ is strictly increasing on $(a,b)$.
(ii) If $f'(x) < 0$ for all $x \in (a,b)$, then $f$ is strictly decreasing on $(a,b)$.

**Proof:** Let us prove (i). Fix any $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. By Theorem 4.2.3, there exists $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0. \quad \Rightarrow f(x_2) - f(x_1) > 0$$

This implies $f(x_1) < f(x_2)$. Therefore, $f$ is strictly increasing on $(a,b)$. The proof of (ii) is similar. $\square$

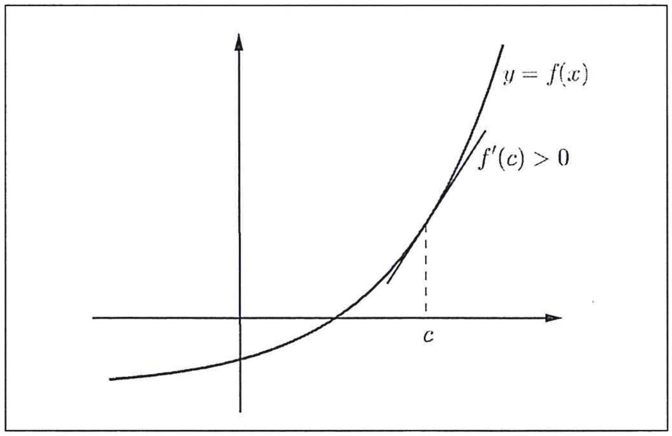



Figure 4.5: Strictly Increasing Function.

**Example 4.3.1** Let $n \in \mathbb{N}$ and $f\colon [0,\infty) \to \mathbb{R}$ be given by $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Therefore, $f'(x) > 0$ for all $x > 0$ and, so, $f$ is strictly increasing. In particular, this shows that every positive real number has exactly one $n$-th root (refer to Example 3.4.2).

**Theorem 4.3.3 — Inverse Function Theorem.** Suppose $f$ is differentiable on $I = (a,b)$ and $f'(x) \neq 0$ for all $x \in (a,b)$. Then $f$ is one-to-one, $f(I)$ is an open interval, and the inverse function $f^{-1}\colon f(I) \to I$ is differentiable. Moreover,

$$(f^{-1})'(y) = \frac{1}{f'(x)}, \quad = \frac{1}{f'(f^{-1}(y))} \tag{4.7}$$

$f$ inc $\Rightarrow f(a,b) = (f(a), f(b))$

$f$ dec $\Rightarrow f(a,b) = (f(b), f(a))$

where $f(x) = y$.

**Proof:** It follows from Theorem 4.2.5 that

$$f'(x) > 0 \text{ for all } x \in (a,b), \text{ or } f'(x) < 0 \text{ for all } x \in (a,b).$$

Suppose $f'(x) > 0$ for all $x \in (a,b)$. Then $f$ is strictly increasing on this interval and, hence, it is one-to-one. It follows from Theorem 3.4.10 and Remark 3.4.11 that $f(I)$ is an open interval and $f^{-1}$ is continuous on $f(I)$.

It remains to prove the differentiability of the inverse function $f^{-1}$ and the representation of its derivative (4.7). Fix any $\bar{y} \in f(I)$ with $\bar{y} = f(\bar{x})$. Let $g = f^{-1}$. We will show that

$$\lim_{y \to \bar{y}} \frac{g(y) - g(\bar{y})}{y - \bar{y}} = \frac{1}{f'(\bar{x})}.$$

$\bar{x} = f^{-1}(\bar{y})$ $\quad f(x_k) = y_k \Rightarrow x_k = f^{-1}(y_k) = g(y_k)$

Fix any sequence $\{y_k\}$ in $f(I)$ that converges to $\bar{y}$ and $y_k \neq \bar{y}$ for every $k$. For each $y_k$, there exists $x_k \in I$ such that $f(x_k) = y_k$. That is, $g(y_k) = x_k$ for all $k$. It follows from the continuity of $g$ that $\{x_k\}$ converges to $\bar{x}$. Then

$$\lim_{k \to \infty} \frac{g(y_k) - g(\bar{y})}{y_k - \bar{y}} = \lim_{k \to \infty} \frac{x_k - \bar{x}}{f(x_k) - f(\bar{x})}$$
$$= \lim_{k \to \infty} \frac{1}{\frac{f(x_k) - f(\bar{x})}{x_k - \bar{x}}} = \frac{1}{f'(\bar{x})}.$$

$f \circ g = id_{f(I)}$ $\quad (f \circ g)(y) = y$

$f'(g(y))\, g'(y) = 1$

$g'(y) = \frac{1}{f'(g(y))}$

$g \circ f = id_I$

$g(f(x)) = x \quad \forall x \in I$

$g'(f(x))\, f'(x) = \frac{dx}{dx} = 1$

$g'(f(x)) = \frac{1}{f'(x)}$

The proof is now complete. □

**Example 4.3.2** Let $n \in \mathbb{N}$ and consider the function $f\colon (0,\infty) \to \mathbb{R}$ given by $f(x) = x^n$. Then $f$ is differentiable and $f'(x) = nx^{n-1} \neq 0$ for all $x \in (0,\infty)$. It is also clear that $f((0,\infty)) = (0,\infty)$. It follows from the Inverse Function Theorem that $f^{-1} : (0,\infty) \to (0,\infty)$ is differentiable and given $y \in (0,\infty)$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{n(f^{-1}(y))^{n-1}}.$$

Given $y > 0$, the value $f^{-1}(y)$ is the unique positive real number whose $n$-th power is $y$. We call $f^{-1}(y)$ the (positive) $n$-th root of $y$ and denote it by $\sqrt[n]{y}$. We also obtain the formula

$$(f^{-1})'(y) = \frac{1}{n(\sqrt[n]{y})^{n-1}}.$$

## Exercises

**4.3.1** (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that if $f'(x)$ is bounded, then $f$ is Lipschitz continuous and, in particular, uniformly continuous.

(b) Give an example of a function $f\colon (0,\infty) \to \mathbb{R}$ which is differentiable and uniformly continuous but such that $f'(x)$ is not bounded.

**4.3.2** ▶ Let $f\colon \mathbb{R} \to \mathbb{R}$. Suppose there exist $\ell \geq 0$ and $\alpha > 0$ such that

$$|f(u) - f(v)| \leq \ell |u - v|^\alpha \text{ for all } u, v \in \mathbb{R}. \tag{4.8}$$

(a) Prove that $f$ is uniformly continuous on $\mathbb{R}$.

(b) Prove that if $\alpha > 1$, then $f$ is a constant function.

(c) Find a nondifferentiable function that satisfies the condition above for $\alpha = 1$.

**4.3.3** ▷ Let $f$ and $g$ be differentiable functions on $\mathbb{R}$ such that $f(x_0) = g(x_0)$ and

$$f'(x) \leq g'(x) \text{ for all } x \geq x_0.$$

Prove that

$$f(x) \leq g(x) \text{ for all } x \geq x_0.$$

**4.3.4** Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions satisfying

(a) $f(0) = g(0) = 1$

(b) $f(x) > 0$, $g(x) > 0$ and $\dfrac{f'(x)}{f(x)} > \dfrac{g'(x)}{g(x)}$ for all $x$.

Prove that

$$\frac{f(1)}{g(1)} > 1 > \frac{g(1)}{f(1)}.$$