Lecture 21: Topological Manifolds, Normal Spaces, Separation Axioms

My aim here is to say just enough for the statements of the major Th^{m}'s of Chapter 7 of Manetti to be understandable.

Def^{n}/ A sub-basis of a topological space is a family $\mathcal{P}$ of open sets such that finite intersections of $\mathcal{P}$ form a basis of the topology.

E1 For the Euclidean topology on $\mathbb{R}$ notice $\mathcal{P} = \{(-\infty, a), (b, \infty) \mid a, b \in \mathbb{R}\}$ serves as sub-basis since $(-\infty, b) \cap (a, \infty) = (a, b)$ for $a < b$ and $\{(a,b)\}$ gives basis for Euclidean Top. on $\mathbb{R}$.

Th^{m}/ (Alexander) Let $\mathcal{P}$ be a sub-basis for X. If every cover of X made by elements of $\mathcal{P}$ has a finite subcover, then X is compact.

Th^{m}/ (Tychonov) The product of an arbitrary family of compact spaces is compact

Th^{m} (7.9) The product of any number of connected spaces is connected

Remark: the topology on infinite products is a bit tricky, I'm not getting into it, but I believe Chapter 7 Manetti has a slightly different approach than Munkres, but I speculate... when I took a course from Munkres I remember lots of box vs. product top...

PARACOMPACTNESS... WHAT'S UP WITH THAT...

Def^n/ A family $\mathcal{A}$ of subsets in a space X is <u>locally finite</u> if every point $x \in X$ admits a nbhd $V \in \mathcal{T}(x)$ such that $V \cap A \neq \emptyset$ for at most finitely many $A \in \mathcal{A}$

For any locally finite family $\{A_i\}$ of subsets, $\overline{\bigcup_i A_i} = \bigcup_i \overline{A_i}$ so the union of locally finite family of closed sets is closed.

Def^n/ Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be covers of X. We say $\{U_i \mid i \in I\}$ <u>refines</u> $\{V_j \mid j \in J\}$ (or is a refinement) if for every $i \in I$ there is a $j \in J$ such that $U_i \subseteq V_j$. In such a case $f : I \to J$ with $U_i \subseteq V_{f(i)}$ for all $i \in I$ is known as a <u>refinement</u> <u>function</u>.

$\boxed{E2}$ The pairwise intersection of two covers $\{U_i \cap V_j \mid (i,j) \in I \times J\}$ is a refinement of both $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$

Def^n/ A topological space is called <u>paracompact</u> if every open cover possesses a locally finite open refinement.

<u>Th$^{\underline{m}}$ (7.15)</u>: Let X be a Hausdorff Space

(1.) If X is exhausted by compact sets, then X is paracompact and locally compact.

(2.) If X is connected and paracompact and locally compact then X has an exhaustion by compact sets.

<u>Corollary 7.16</u>: Every locally compact, 2$^{\underline{nd}}$ countable Hausdorff space is paracompact.

Th$^{\underline{m}}$ (Stone) : Every metrizable space is paracompact

See §7.3 of Manetti for proof and/or reference.

Topological Manifolds (§ 7.4)

Defⁿ/ A space M is called an <u>n-dimensional</u> <u>topological</u> <u>manifold</u> if

(1.) M is Hausdorff

(2.) every pt. in M has an open nbhd homeomorphic to an open set of $\mathbb{R}^n$

(3.) every connected component of M is 2^{nd} countable

can replace with paracompactness (Cor 7.29)

$\boxed{E3}$ open $U \subseteq \mathbb{R}^n$ is an n-dim'l manifold.
any open $U \subseteq \mathbb{C}^n$ is a $2n$-dim'l manifold since $\mathbb{C}^n = \mathbb{R}^{2n}$

$\boxed{E4}$ The sphere S^n is a topological manifold of dimension n
Notice $x \in S^n - \{-x\} \cong \mathbb{R}^n$ via stereographic projection.
(see page 13 of Manetti!)

$\boxed{E5}$ Real projective space $P^n(\mathbb{R})$ is an n-dim'l manifold. Each pt. lies in complement of some hyperplane H and $P^n(\mathbb{R}) - H \cong \mathbb{R}^n$ (Example 5.21 on pg. 96 gives homeomorphism $(f^{-1} \circ \pi)(x_0, \ldots, x_n) = f^{-1}[x_0, \ldots, x_n] = \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in \mathbb{R}^n$)

$\boxed{E6}$ $P^n(\mathbb{C})$ is $2n$-dim'l topological manifold

$\boxed{E7}$ V a finite dim'l vect. space over $\mathbb{R}$ is a <u>linear</u> <u>manifold</u>.

Proposition 7.27: A topological manifold is locally compact and Hausdorff. Each connected component is exhausted by compact sets.

Corollary 7.28: Every topological manifold is paracompact.

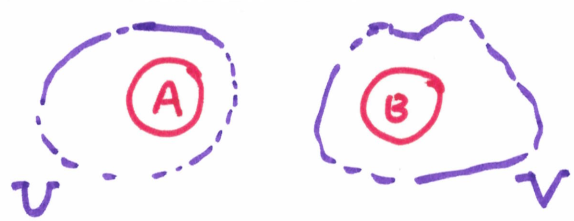
Corollary 7.29: Let M be paracompact, Hausdorff and such that any open nbhd homeomorphic to an open set in $\mathbb{R}^n$. Then M is a topological manifold.

NORMAL SPACES (§7.5)

Def

Def^{n}/ A space is said to be <u>NORMAL</u> if it is Hausdorff and disjoint closed sets are <u>separated</u> by disjoint open sets

$$\underbrace{A \qquad\qquad B}_{\text{disjoint, closed sets}}$$



$A \subseteq U$
$B \subseteq V$
$U \cap V = \emptyset$

Proposition (7.31)

(1.) Every metrizable space is normal.
(2.) Every paracompact Hausdorff space is normal.

SEPARATION AXIOMS (§7.6)

Terminology introduced by Tietze in 1921 as the Trennbarkeitsaxiome

Def^{n}/ A space is called:

T0 : if distinct points have distinct closures

T1 : if every point is closed

T2 : if points $c \neq d$ are contained in disjoint open sets; $c \in U$, $d \in V$ and $U \cap V = \emptyset$ (Hausdorff)

T3 : if every closed set C and every point $d \notin C$ are contained in disjoint open sets; $C \subseteq U$, $d \in V$ and $U \cap V = \emptyset$

T4 : if every two closed disjoint sets are contained in disjoint open sets; $C \subseteq U$, $D \subseteq V$ and $U \cap V = \emptyset$

$\boxed{\text{Def}^{n}/\text{ A space is called REGULAR if it is T1 and T3}}$

$\boxed{\text{NORMAL} = \text{T1 and T4}}$

Remark: metrizable $\Rightarrow$ normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff $\Rightarrow$ T1 $\Rightarrow$ T0

Th^{m}/ Every 2^{nd} countable regular space is normal. In particular, subspaces and finite products of 2^{nd} countable normal spaces are normal.

$\boxed{\text{Th}^{m}\text{ (URYSOHN) EVERY } 2^{nd} \text{ COUNTABLE NORMAL SPACE IS METRIZABLE}}$