LECTURE 22 : HOMOTOPY

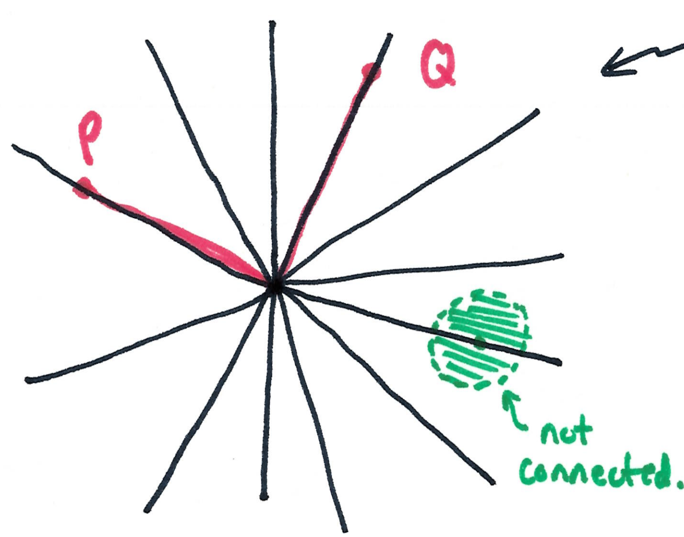
(Chapter 10, Manetti)

We tend to sell topology to nontechnical audiences with homotopy. In particular, a homotopy is roughly a formal method of implementing one shape flexing into another. It has to do with the deformation of one shape (like a donut) and another (like a coffee mug). I'll follow Manetti, so it will be a few minutes until we get to the deformation...

Def^n/ A space is <u>locally</u> <u>connected</u> if every point has local basis of connected neighbourhoods

Connected components of locally connected space are open, In general connected spaces may fail to be locally connected

$\boxed{\text{E1}}$ $X \subseteq \mathbb{R}^2$ formed by union of all lines $ax = by$, $a, b \in \mathbb{Z}$ not both zero forms a path connected, but not locally path connected space. (Ex 10.1 in Manetti, p. 168)



X contains <u>all</u> lines through $(0,0)$ with rational slope, it's dense in $\mathbb{R}^2$ I'm fairly certain. Anyway, it's <u>path-connected</u> $\Rightarrow X$ connected. YET, does not have local basis of connected nbhds.

Def^{n}/ Let X be a topological space. Denote $\pi_0(X) = X/\sim$ the quotient space w.r.t. relation $\sim$ which is defined by $x \sim y$ iff there exists <u>path</u> in X from x to y.

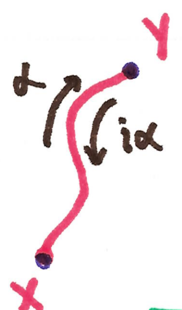
The equivalence classes of $\sim$ are the connected components of X.

Def^{n}/ $\Omega(X, x, y) = \{\alpha : [0,1] \to X \mid \alpha \text{ continuous}, \alpha(0) = x, \alpha(1) = y\}$ is the set of <u>paths</u> from x to y in X.

Let's pause to reflect on details of $\sim$, this sets-up notation for future work with paths, note $x \sim y \iff \Omega(X, x, y) \neq \emptyset$.

- <u>Reflexive</u>: to see $x \sim x$ notice $I_x : [0,1] \to X$ by $I_x(t) = x \;\; \forall t \in [0,1]$ is continuous map thus $\Omega(X, x, x) \ni I_x$ and $x \sim x$ is clear.

- <u>SYMMETRY</u>: for every $x, y \in X$ for which $x \sim y$ we have $\exists\, \alpha : [0,1] \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$. Notice $\gamma(t) = \alpha(1-t)$ defines $\gamma : [0,1] \to X$ continuous with starting point $y = \gamma(0) = \alpha(1)$ and ending or terminal pt. $x = \gamma(1) = \alpha(0)$. Thus $\gamma \in \Omega(X, y, x) \;\therefore\; y \sim x$.

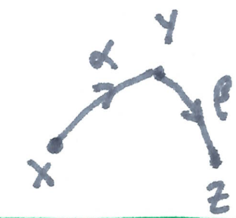


Def^{n}/ $i : \Omega(X, x, y) \to \Omega(X, y, x)$ is the <u>path-reversal</u> operator

- **TRANSITIVTY**: $x \sim y$ and $y \sim z \Rightarrow x \sim z$ is true by the following

Def^{n}/ product of paths $* : \Omega(X, x, y) \times \Omega(X, y, z) \longrightarrow \Omega(X, x, z)$ denoted by $(\alpha, \beta) \longmapsto \alpha * \beta$ where we define

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le 1/2 \\ \beta(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

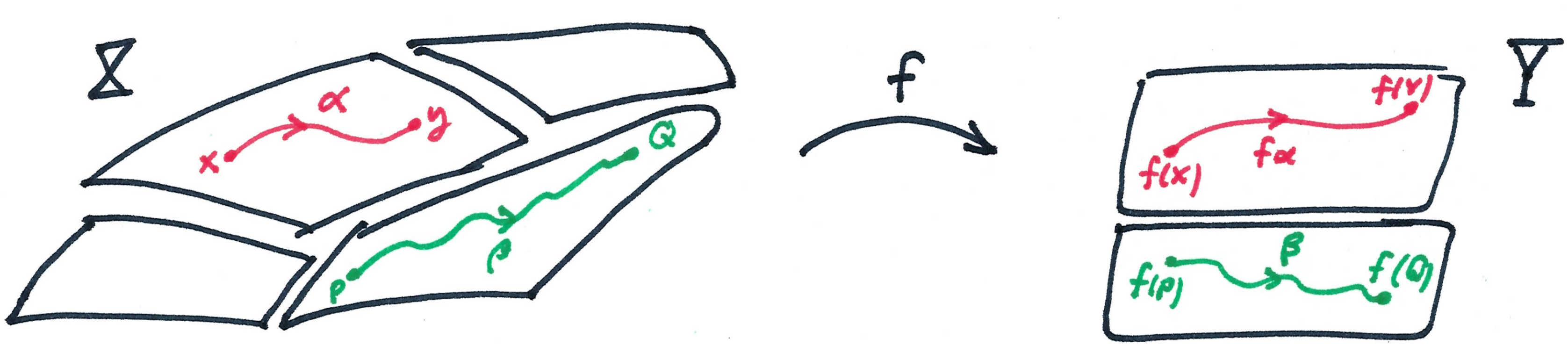


Remark: given path $\alpha : [a,b] \longrightarrow X$ we can reparametrize to $\tilde{\alpha} \in \Omega(X, x, y)$ where $x = \alpha(a)$ and $y = \alpha(b)$ by $\tilde{\alpha}(t) = \alpha((1-t)a + tb)$ for $0 \le t \le 1$.

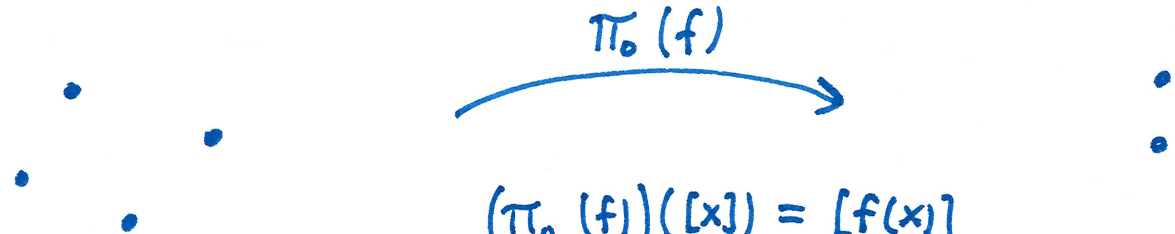
Equivalence classes for $\sim$ are <u>path components</u> of X. A space is called <u>locally path connected</u> if any point has local basis of path connected neighbourhoods. For example, $B(x,t)$, $0 < t < r$ give local path connected basis for $\mathbb{R}^n$.

PROPOSITION: Suppose X is locally path connected. Then the path components of X coincide with the connected components of X

Next we discuss how to transport paths via a map $f : X \rightarrow Y$



$\pi_0(X) = \dfrac{X}{\sim}$ $\qquad (\pi_0(f))([\alpha]) = [f \circ \alpha]$ $\qquad \pi_0(Y) = \dfrac{Y}{\sim}$



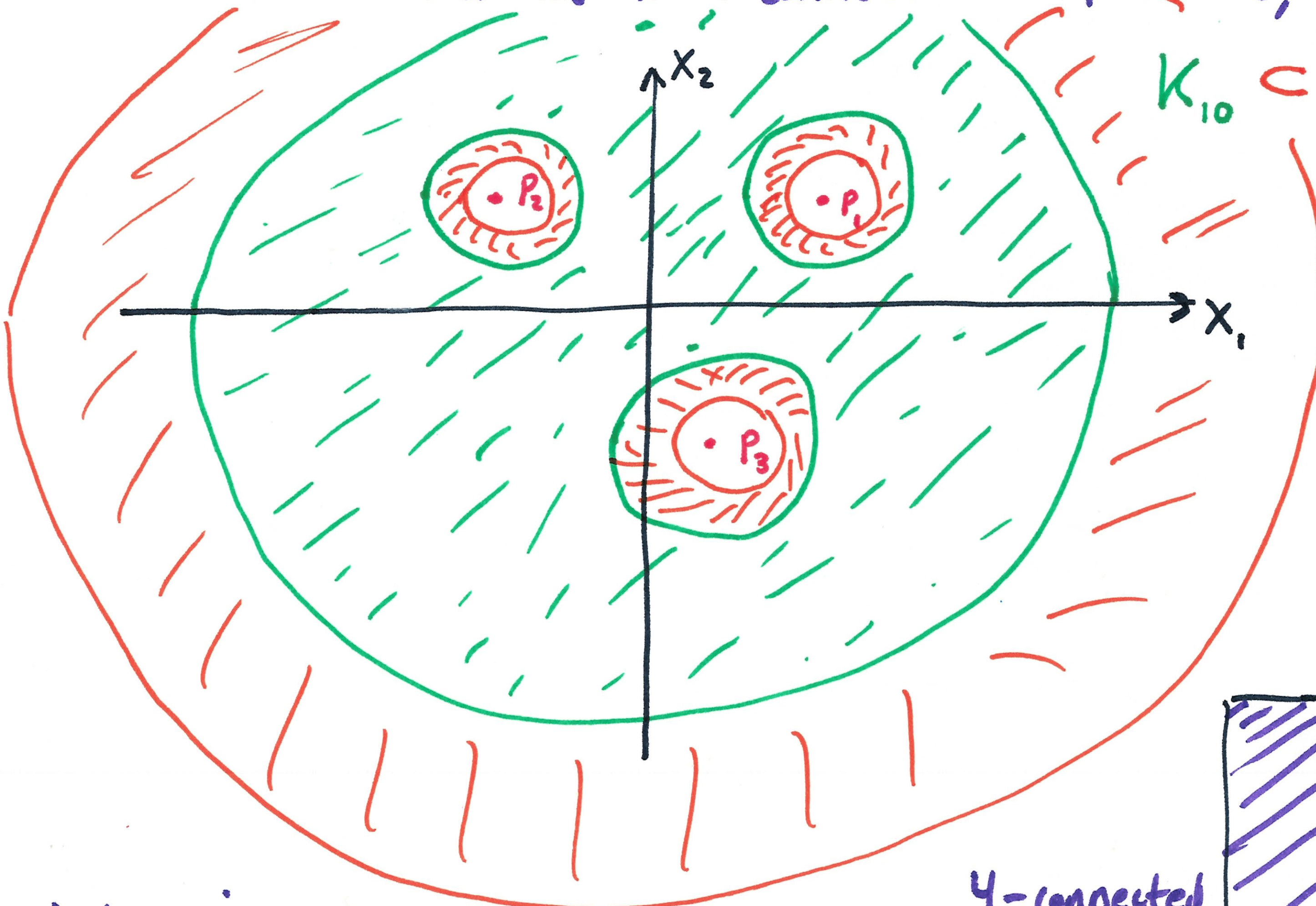
$(\pi_0(f))([x]) = [f(x)]$

$[x] \xmapsto[\pi_0(f)]{} [f(x)]$

π_0 is a <u>functor</u> from the category of topological spaces to the category of sets. It sends identity to identity and interfaces with composition in the sence if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous and $g \circ f : X \to Z$ then $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$. (Manetti completes this discussion in §10.4)

$\boxed{E2}$ $K_n = \{ x \in \mathbb{R}^m / \ \|x\| \leq n, \ \|x - P_i\| \geq \frac{1}{n} \}$

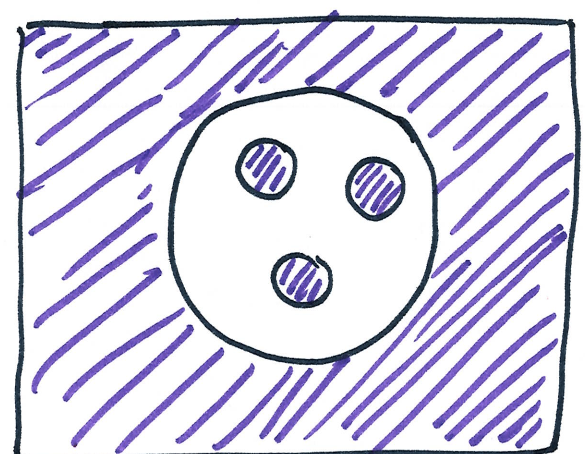
Consider $X = \mathbb{R}^m - \{P_1, P_2, \ldots, P_t\}$. When $n >> 0$ we can see $X - K_n$ has $t+1$ connected components,



$K_{10} \subset K_{20}$

inclusion map is injective & induces bijection $\pi_0(K_{10}) \to \pi_0(K_{20})$ etc..

$X - K_n$



4-connected components

inclusion $i : X - K_{n+1} \longrightarrow X - K_n$ induces bijection of $\pi_0(X - K_{n+1})$ and $\pi_0(X - K_n)$ $\Rightarrow$ cardinality of $\pi_0(X - K_{n+1})$ is a topological invariant of X (by Example 10.6)

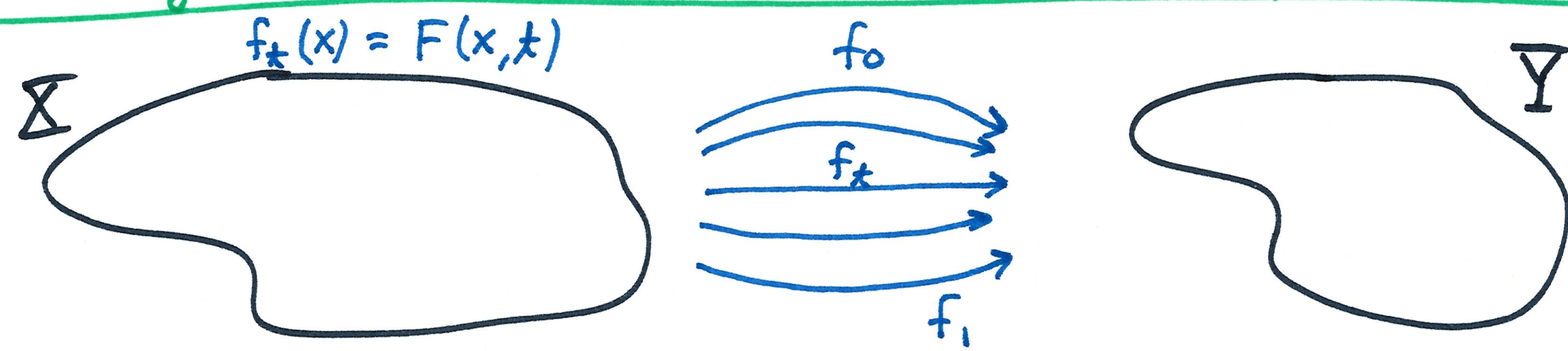
<u>Remark</u>: we can see $\mathbb{R}-S$ and $\mathbb{R}-T$ for finite sets S & T are not homeomorphic if $|S| \neq |T|$. $\boxed{E2}$ generalizes that problem on your Final Exam to $\mathbb{R}^m - S$ where $S = \{p_1, \dots, p_t\}$. Since $t+1$ is <u>topological</u> <u>invariant</u> of $\mathbb{R}^m - S$ it follows that $\mathbb{R}^m - T$ homeomorphic to $\mathbb{R}^m - S$ must also have $t+1$ connected comp. for the compact exhaustion, and that fails if $|S| \neq |T|$.
(Example 10.6 shows cardinality of $\pi_0(X - K_n)$ is topological invariant)

§10.2 HOMOTOPY

Def^n/ Two continuous maps $f_0, f_1 : X \to Y$ are <u>homotopic</u> if $\exists$ continuous $F : X \times [0,1] \to Y$ such that

(1.) $F(x,0) = f_0(x)$
(2.) $F(x,1) = f_1(x)$

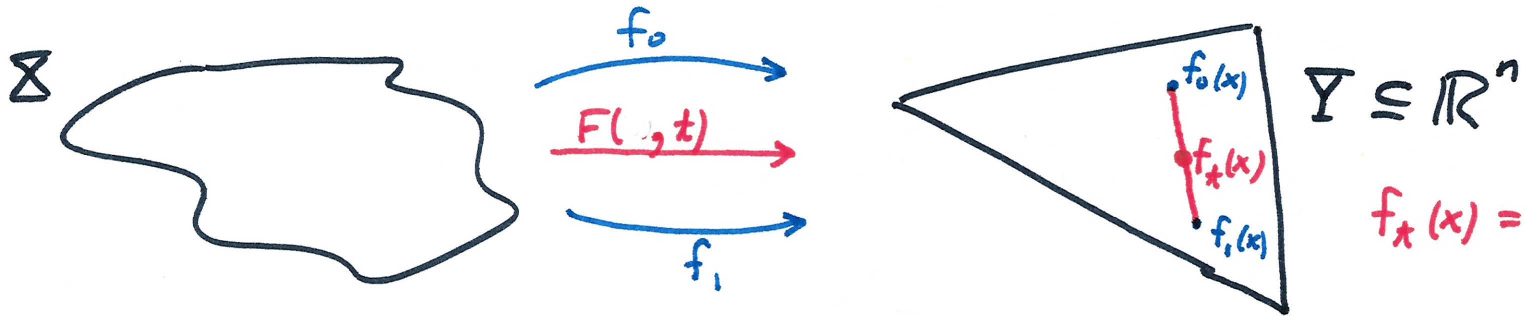
for every $x \in X$. Such an F is called a homotopy between f_0 & f_1



$\boxed{E3}$ A convex subset $Y \subseteq \mathbb{R}^n$ has, for any topological space X maps $f_0, f_1 : X \to Y$ homotopic via:

$$F(x,t) = (1-t)f_0(x) + t f_1(x)$$

where $F : X \times [0,1] \to Y$ is a continuous map.



$f_t(x) = F(x,t)$.

$\boxed{E4}$ $f, g : X \to \mathbb{R}^n - \{0\}$ where $\|f(x) - g(x)\| < \|f(x)\|$ for each $x \in X$.

Then we can show f and g homotopic via the homotopy

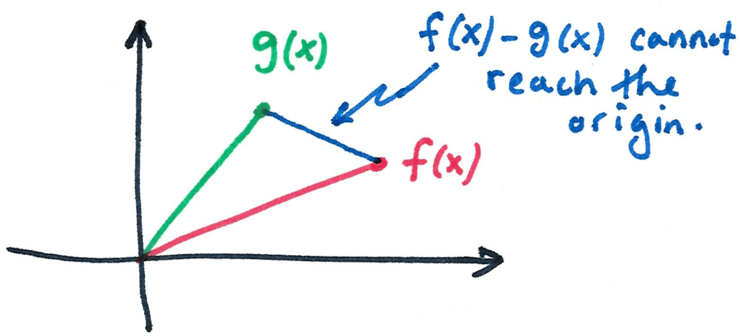
$$F(x,t) = (1-t)f(x) + t g(x)$$
$$= f(x) - t(f(x) - g(x))$$

Suppose $f(x) - t(f(x) - g(x)) = 0 \Rightarrow f(x) = t[f(x) - g(x)]$

$\Rightarrow \|f(x)\| = t\|f(x) - g(x)\| \le \|f(x) - g(x)\|$

$\uparrow$ if $0 \le t \le 1$

$\rightarrow\leftarrow$ the given $\|f(x) - g(x)\| < \|f(x)\|$



Lemma: Homotopy provides an equivalence relation on $C(X, Y)$, the set of continuous maps from topological space X to top space Y

Proof: Let $f: X \to Y$ be continuous then $F: X \times I \to Y$ defined by $F(x,t) = f(x)\ \forall (x,t) \in X \times I$ shows $f \sim f$ that is f is homotopic to f thus homotopy is reflexive. Suppose $f \sim g$ by $F: X \times I \to Y$ where $F(x,0) = f(x)$ & $F(x,1) = g(x)$ then $G(x,t) = F(x, 1-t)$ gives homotopy for $g \sim f$ $\therefore \sim$ symmetric. Finally, if $f \sim g$ by F and $g \sim h$ by G we construct homotopy H for $f \sim h$ via $H: X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t) & : 0 \leq t \leq \tfrac{1}{2} \\ G(x, 2t-1) & : \tfrac{1}{2} \leq t \leq 1 \end{cases}$$

$\boxed{E5}$ The map $f: S^n \to S^n$ given by $f(x) = -x$ is the antipodal map. If $n = 2k-1$ then $S^n = \{ z \in \mathbb{C}^k \mid \|z\| = 1 \}$ and consequently $F: S^n \times I \to S^n$ where $F(x,t) = \exp(\pi i t)x$ gives $F(x,0) = x = Id(x)$ and $F(x,1) = e^{\pi i}x = -x = f(x)$ thus $f \sim Id$.

Remark: the homotopy seen in $\boxed{E5}$ is not possible when n is even. Manetti mentions this is beyond the text on p. 170.

<u>Lemma</u>: Suppose we have continuous f_0, g_0, f_1, g_1 as

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

If $f_0 \sim f_1$ and $g_0 \sim g_1$ then $g_0 \circ f_0 \sim g_1 \circ f_1$

Def^n/ A continuous map $f : X \to Y$ is a <u>homotopy equivalence</u> if there exists a continuous map $g : Y \to X$ such that $f \circ g \sim Id_Y$ and $g \circ f \sim Id_X$. In this case we say X & Y are <u>homotopy equivalent</u>.

If $f : X \to Y$ is a homeomorphism then $f \circ f^{-1} = Id_Y \sim Id_Y$ and $f^{-1} \circ f = Id_X \sim Id_X$ $\therefore$ X and Y are homotopic. The converse fails: X and Y may be homotopic and yet not homeomorphic.

$\boxed{E6}$ Suppose $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are convex subsets where $n \neq m$ then $f : X \to Y$ given by $f(x) = y_0$ $\forall x \in X$ for some $y_0 \in Y$ and $g : Y \to X$ given by $g(y) = x_0$ $\forall y \in Y$ for some $x_0 \in X$. Then applying the homotopy of $\boxed{E3}$ note $f \circ g \sim Id_Y$ and $g \circ f \sim Id_X$ thus X and Y are homotopic.

(But, $X \cong \mathbb{R}^n \not\cong \mathbb{R}^m \cong Y$)

Lemma: If two continuous maps $f, g : X \to Y$ are homotopic, then $\pi_0(f) = \pi_0(g) : \pi_0(X) \to \pi_0(Y)$. Moreover, $\pi_0(f)$ is a bijection if $f : X \to Y$ is a homotopy equivalence.

Proof: recall $\pi_0(f) : X/\sim \;\to Y/\sim\; = \pi_0(Y)$ was the map $[x] \longmapsto [f(x)]$ and $(\pi_0(g))[x] = [g(x)]$. But, $f \sim g$ gives $H : X \times I \to Y$ where $H(x,0) = f(x)$ & $H(x,1) = g(x)$ and we note H allows us to form a path from $f(x)$ to $g(x)$ via $\alpha(t) = H(x,t)$. Thus $[f(x)] = [g(x)] \Rightarrow \pi_0(f) = \pi_0(g)$. Next, if $\exists\, \gamma : Y \to X$ continuous and serving as homotopy inverse for f; $f \circ \gamma \sim Id_Y$ and $\gamma \circ f \sim Id_X$

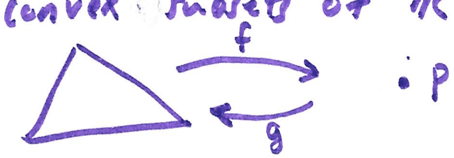
$$\pi_0(f)\,\pi_0(\gamma) = \pi_0(Id_Y) = 1_{\pi_0(Y)}$$
$$\pi_0(\gamma)\,\pi_0(f) = \pi_0(Id_X) = 1_{\pi_0(X)}$$

Def^n/ A space X is <u>contractible</u> if X is homotopic to a point.

If X homotopic to $\{p\}$ then $\exists\, f : X \to \{p\}$ and $g : \{p\} \to X$ for which $f \circ g \sim Id_{\{p\}}$ and $g \circ f \sim Id_X$. Notice $(g \circ f)(x) = g(f(x)) = p$ thus $g \circ f$ is a constant map $\Rightarrow$ constant map homotopic to identity map on X

(Ex. 10.6 #1 on p. 172 is half solved here)

$\boxed{E7}$ All convex subsets of $\mathbb{R}^n$ are contractible following argument like that in $\boxed{E3}$



Contractible $\Rightarrow$ path connected

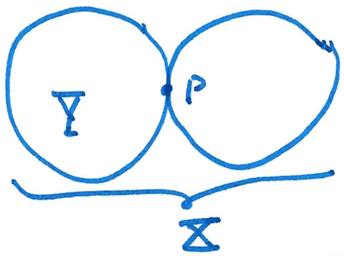
However, path connected $\not\Rightarrow$ contractible (see S^1)

Def^n/ A subspace $Y \subseteq X$ is a <u>retract</u> of X if there is a continuous map $r : X \to Y$ such that $r(y) = y$ $\forall y \in Y$

$\boxed{E8}$ $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \{p\}$ then if

$$r(x) = \begin{cases} x & \text{if } x \in X_1 \\ p & \text{if } x \in X_2 \end{cases}$$

is continuous then X_1 is retract of $X_1 \cup X_2$.



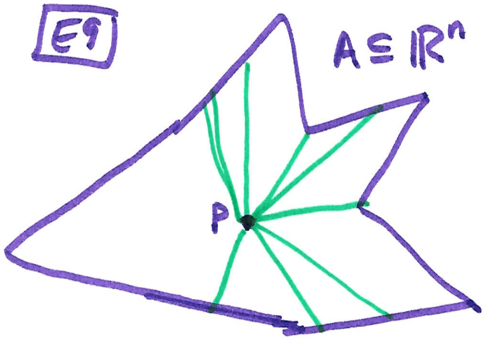
has $Y \subseteq X$ where Y is <u>retract</u> of X.

Def^n/ A subspace $Y \subseteq X$ is called a <u>deformation retract</u> of X if there is a continuous map $R : X \times [0,1] \to X$ called a <u>deformation</u> of X into Y such that

(1.) $R(x,0) \in Y$ and $R(x,1) = x$ for all $x \in X$,

(2.) $R(y,t) = y$ for every $y \in Y$ and $t \in [0,1]$.

$\boxed{E9}$



$A \subseteq \mathbb{R}^n$ star-shaped with star center p

$R : A \times [0,1] \to A$

$R(x,t) = tx + (1-t)p$

$R(x,0) = p$

$R(y,t) = ty + (1-t)p$
$= tp + p - tp$
$= p$ (y = p only case here)

Proposition: A deformation retract $Y \subseteq X$ is a retract of X and the inclusion $i : Y \hookrightarrow X$ is a homotopy equivalence

Proof: Suppose $R : X \times [0,1] \to X$ gives the deformation of X to Y then $r : X \to Y$ defined by $R(x,0) = i(r(x))$, and R serves as homotopy between $i \circ r$ and Id_X. Note $r \circ i = Id_Y$ thus i and r are homotopy equivalences.

Def^n (1.) $R(x,0) \in Y$ and $R(x,1) = x \quad \forall x \in X$
(2.) $R(y,t) = y \quad \forall y \in Y,\ t \in I.$

$Y \subseteq X$ ↑ deformation retract of X

$\boxed{E10}$ S^n is deformation retract of $\mathbb{R}^{n+1} - \{0\}$ via the deformation $R : (\mathbb{R}^{n+1} - \{0\}) \times I \longrightarrow \mathbb{R}^{n+1} - \{0\}$ given by

$$R(x,t) = tx + (1-t)\frac{x}{\|x\|}$$

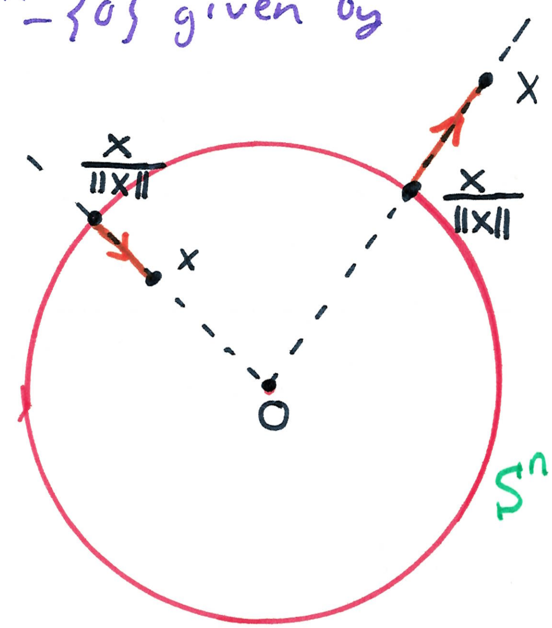
$$R(x,0) = \frac{x}{\|x\|} \in S^n \text{ for all } x \neq 0$$

$$R(x,1) = x \quad \forall x \neq 0$$

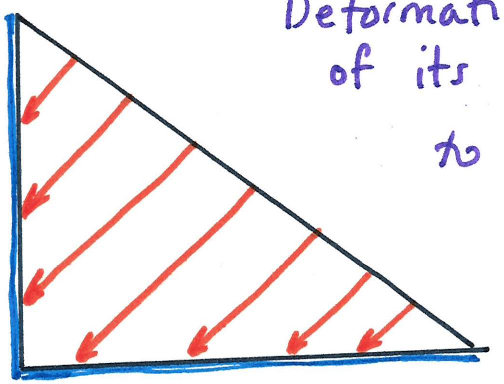
$$\underbrace{R(y,t) = ty + (1-t)\frac{y}{\|y\|}}$$

for fixed y this parametrizes line-segment path from

$$\underset{(t=0)}{\frac{y}{\|y\|}} \longrightarrow \underset{(t=1)}{y}$$



E11



Deformation of triangle to two of its sides, Manetti generalizes to any non-degenerate simplex $\Delta \subseteq \mathbb{R}^n$ deforms to m of its faces for every $1 \le m \le n$.

Remark: reading §10.4 is a good idea. Category theory is likely part of your graduate school education.

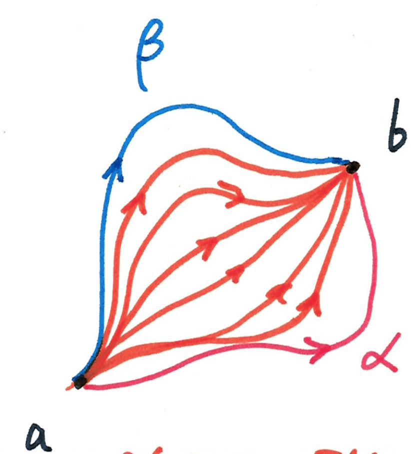
PATH HOMOTORY (§11.1)

Def^n/ Two paths $\alpha, \beta \in \Omega(X, a, b)$ are path homotopic if there exists a continuous map $F : I \times I \longrightarrow X$ such that

(1.) $F(t, 0) = \alpha(t)$, $F(t, 1) = \beta(t)$

(2.) $F(0, s) = a$, $F(1, s) = b$ $\forall s \in I$

and such a map as F is called a path homotopy between α and β



$\gamma_s(t) = F(t, s)$

$\gamma_s(0) = F(0, s) = a$

$\gamma_s(1) = F(1, s) = b$

- We write $\underbrace{\alpha \sim \beta}$ when α & β are path homotopic

gives equivalence relation as before.

E12 Suppose $X \subseteq \mathbb{R}^n$ is convex and $\alpha, \beta \in \Omega(X, a, b)$ then $F(t,s) = s\beta(t) + (1-s)\alpha(t)$ shows $\alpha \sim \beta$

$$F(t,0) = \alpha(t) \qquad F(0,s) = s\beta(0) + (1-s)\alpha(0) = sa + (1-s)a = a$$
$$F(t,1) = \beta(t) \qquad F(1,s) = s\beta(1) + (1-s)\alpha(1) = sb + (1-s)b = b$$

path homotopy is a little boring in convex space

Th^{m}/ Composition and inversion of paths commute with path equivalence.

(1.) For $\alpha, \alpha' \in \Omega(X, a, b)$ and $\beta, \beta' \in \Omega(X, b, c)$ if $\alpha \sim \alpha'$ and $\beta \sim \beta'$ then $\alpha * \beta \sim \alpha' * \beta'$

(2.) For $\alpha, \alpha' \in \Omega(X, a, b)$ and $\alpha \sim \alpha'$ we find $\underbrace{i(\alpha) \sim i(\alpha')}_{\text{inverse, or reversal of path.}}$

Lemma 11.3: Let $\alpha : I \to X$ be a path and $\phi : I \to I$ any continuous map s.t. $\phi(0) = 0$, $\phi(1) = 1$ then $\alpha(t)$ is path equivalent to $\beta(t) = \alpha(\phi(t))$

Proof: $F(t,s) = \alpha(s\phi(t) + (1-s)t)$ has
$$F(t,0) = \alpha(t)$$
$$F(t,1) = \alpha(\phi(t))$$
$$F(0,s) = \qquad = \alpha(0)$$
$$F(1,s) = \alpha(s\phi(1) + (1-s)1) = \alpha(1)$$

PROPOSITION 11.4: $\alpha * (\beta * \gamma) \sim (\alpha * \beta) * \gamma$
for any $\alpha \in \Omega(X, a, b)$, $\beta \in \Omega(X, b, c)$ and $\gamma \in \Omega(X, c, d)$
thus the homotopy class $\alpha * \beta * \gamma$ is well-defined.

PROOF: $((\alpha * \beta) * \gamma)(t) = (\alpha * (\beta * \gamma))(\phi(t))$ where $\phi(t) = \begin{cases} 2t & 0 \le t \le 1/4 \\ t + \frac{1}{4} & \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{t+1}{2} & \frac{1}{2} \le t \le 1 \end{cases}$

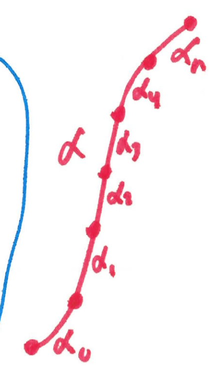
PROPOSITION 11.6: For any path $\alpha \in \Omega(X, a, b)$

(1.) $1_a * \alpha \sim \alpha * 1_b \sim \alpha$

(2.) $\alpha * i(\alpha) \sim 1_a$

where 1_a and 1_b are constant paths a & b respectively

Corollary 11.8: Let $\alpha : I \to X$ be a path and $P_1, \ldots, P_n \in [0,1]$ a given set of points. Setting $P_0 = 0$, $P_{n+1} = 1$ and letting α_i be standard parametrization of restriction of α to $[P_i, P_{i+1}]$ (that is $\alpha_i(t) = \alpha((1-t)P_i + t P_{i+1})$) then $\alpha \sim \alpha_0 * \alpha_1 * \cdots * \alpha_n$



Prop. 11.9: Let $f : X \to Y$ be a continuous mapping

(1.) $\alpha, \beta \in \Omega(X, a, b)$, if $\alpha \sim \beta$ then $f \circ \alpha \sim f \circ \beta$

(2.) $\alpha \in \Omega(X, a, b)$ and $\beta \in \Omega(X, b, c)$. Then $f(\alpha * \beta) = (f \circ \alpha) * (f \circ \beta)$ and $i(f \circ \alpha) = f(i(\alpha))$.

THE FUNDAMENTAL GROUP (§11.2)

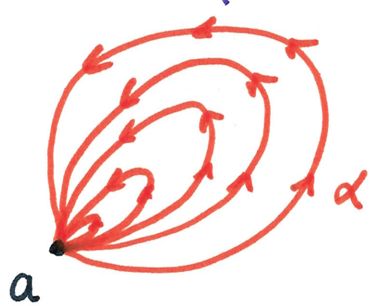
Def^n/ Given a space X and a point $a \in X$ we define the <u>fundamental group</u> or <u>first homotopy group</u> of X with base pt. a as

$$\frac{\Omega(X,a,a)}{\sim} = \{ [\alpha] \mid \alpha \in \Omega(X,a,a) \} = \pi_1(X,a)$$

where $[\alpha] = \{ \alpha' \mid \alpha \sim \alpha' \text{ for some } \alpha' \in \Omega(X,a,a) \}$. We call $\Omega(X,a,a)$ the set of loops based at a. The group structure of $\Omega(X,a,a)/\sim$ is denoted by juxtaposition,

$$[\alpha][\beta] = [\alpha * \beta]$$
$$[\alpha]^{-1} = [\, i(\alpha) \,]$$

$\boxed{\text{E13}}$ Let $X \subseteq \mathbb{R}^n$ be convex subspace then for any $a \in X$ we find $\pi_1(X,a) = 0$. Observe, for loop $\alpha \in \Omega(X,a,a)$ we have path homotopy $F(t,s) = sa + (1-s)\alpha(t)$

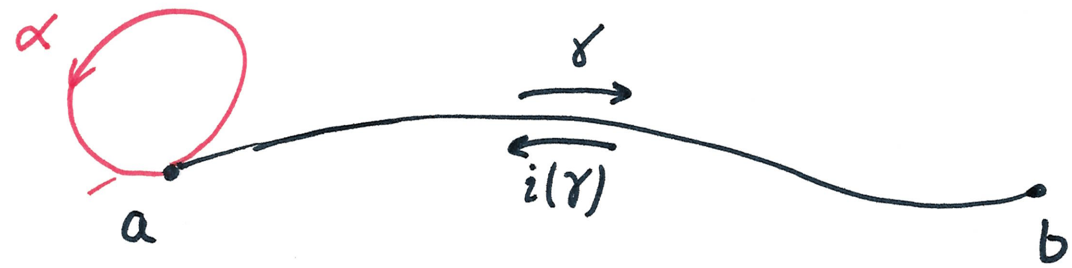


$$F(t,0) = \alpha(t)$$
$$F(t,1) = a = 1_a(t)$$
$$F(0,s) = sa + (1-s)\alpha(0) = sa + (1-s)a = a$$
$$F(1,s) = sa + (1-s)\alpha(1) = sa + (1-s)a = a.$$

Lemma 11.13: Take $\gamma \in \Omega(X, a, b)$ and define

$$\gamma_\# : \pi_1(X, a) \longrightarrow \pi_1(X, b)$$

by $\gamma_\#[\alpha] = [i(\gamma) * \alpha * \gamma]$. Then $\gamma_\#$ is a group homomorphism



Remark: the fundamental group at any two pts in a path component will match. Thus, if X is path connected then,

$$\begin{aligned}\gamma_\#[\alpha]\,\gamma_\#[\beta] &= [i(\gamma) * \alpha * \gamma][i(\gamma) * \beta * \gamma] \\ &= [i(\gamma) * \alpha * \underbrace{(\gamma * i(\gamma))}_{1_a} * \beta * \gamma] \\ &= [i(\gamma) * \alpha * \beta * \gamma] \\ &= \gamma_\#[\alpha * \beta].\end{aligned}$$

$i(i(\gamma)) = \gamma$

$$\begin{aligned}(i(\gamma)_\#)(\gamma_\#[\alpha]) &= [\gamma * \quad \gamma_\#[\alpha] * i(\gamma)] \\ &= [\gamma * \quad i(\gamma) * \alpha * \gamma * i(\gamma)] \\ &= [1_a * \alpha * 1_a] \\ &= [\alpha] \quad \therefore \quad (\gamma_\#)^{-1} = (i(\gamma))_\#.\end{aligned}$$

Def^{n}/ $\pi_1(X)$ is isomorphism class of its fundamental group w.r.t. any base point

Def^n/ A space is <u>simply</u> <u>connected</u> if it is <u>path</u> <u>connected</u> and has <u>trivial</u> fundamental group ($\pi_1(X) = 0$)

E14 Let $X \subseteq \mathbb{R}^n$ be convex then $\pi_1(X,a) = 0 \ \forall a \in X$ thus $\pi_1(X) = 0$ (X is path connected)

$\boxed{\pi_1(\mathbb{R}^n) = 0}$

E15 Let $n \in \mathbb{Z}$ and $\alpha_n : [0,1] \longrightarrow S^1$ be path $\alpha_n(t) = \exp(2\pi i n t) = \cos(2\pi n t) + i\sin(2\pi n t)$. Then we can prove $\mathbb{Z} \xmapsto{\psi} \pi_1(S^1, 1)$ given by $\psi(n) = [\alpha_n]$ is a group homomorphism, well, later Manetti proves it's an isomorphism.

$\boxed{\pi_1(S^1) = \mathbb{Z}}$

$$[\alpha_0] = [1_1] \ , \ i\alpha_n = \alpha_{-n} \ , \ \alpha_{n+m} \sim \alpha_n * \alpha_m$$

<u>Proposition 11.17</u>: The fundamental group of a product of two spaces is isomorphic to the product of the fund. groups;

$$\pi_1(X \times Y, (a,b)) = \pi_1(X,a) \times \pi_1(Y,b)$$

E16 $\pi_1(S^1_1 \times S^1_1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.