(c) $\lim_{x\to\infty} \dfrac{\frac{\pi}{2}-\arctan x}{\ln(1+\frac{1}{x})}$.

(d) $\lim_{x\to\infty} \sqrt{x}e^{-x}$. (*Hint:* first rewrite as a quotient.)

4.4.4 Prove that the following functions are differentiable at 1 and -1.

(a) $f(x) = \begin{cases} x^2e^{-x^2}, & \text{if } |x| \le 1; \\ \dfrac{1}{e}, & \text{if } |x| > 1. \end{cases}$

(b) $f(x) = \begin{cases} \arctan x, & \text{if } |x| \le 1; \\ \dfrac{\pi}{4}\operatorname{sign} x + \dfrac{x-1}{2}, & \text{if } |x| > 1. \end{cases}$

4.4.5 ▷ Let $P(x)$ be a polynomial. Prove that

$$\lim_{x\to\infty} P(x)e^{-x} = 0.$$

4.4.6 ▶ Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that $f \in C^n(\mathbb{R})$ for every $n \in \mathbb{N}$.

LECTURE 22: TAYLOR'S THEOREM

4.5 TAYLOR'S THEOREM

In this section, we prove a result that lets us approximate differentiable functions by polynomials.

Theorem 4.5.1 — Taylor's Theorem. Let n be a positive integer. Suppose $f\colon [a,b] \to \mathbb{R}$ is a function such that $f^{(n)}$ is continuous on $[a,b]$, and $f^{(n+1)}(x)$ exists for all $x \in (a,b)$. Let $\bar{x} \in [a,b]$. Then for any $x \in [a,b]$ with $x \neq \bar{x}$, there exists a number c in between $\bar{x}$ and x such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-\bar{x})^{n+1},$$

where

n^{th} Taylor Poly, centered at $\bar{x}$ for $f(x)$.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\bar{x})}{k!}(x-\bar{x})^k. = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{1}{2}f''(\bar{x})(x-\bar{x})^2 + \cdots + \frac{f^{(n)}(\bar{x})}{n!}(x-\bar{x})^n$$

Proof: Let $\bar{x}$ be as in the statement and let us fix $x \neq \bar{x}$. Since $x - \bar{x} \neq 0$, there exists a number $\lambda \in \mathbb{R}$ such that

$$f(x) = P_n(x) + \frac{\lambda}{(n+1)!}(x-\bar{x})^{n+1}.$$

(Claim) $\dfrac{\big(f(x) - P_n(x)\big)(n+1)!}{(x-\bar{x})^{n+1}} \underset{\text{def}^{\text{n}}}{=} \lambda$

We will now show that

$$\lambda = f^{(n+1)}(c),$$

$P_n(x)$ with $\bar{x} = t$

$P_n^t(x)$ vs. $P_n^{\bar{x}}(x)$

for some c in between $\bar{x}$ and x.

Consider the function

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k - \frac{\lambda}{(n+1)!}(x-t)^{n+1}.$$

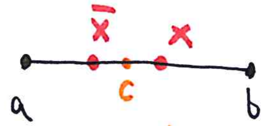
(Def$^{\underline{n}}$ of g)

Then

$$g(\bar{x}) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(\bar{x})}{k!}(x-\bar{x})^k - \frac{\lambda}{(n+1)!}(x-\bar{x})^{n+1} = f(x) - \underbrace{P_n(x) - \frac{\lambda}{(n+1)!}(x-\bar{x})^{n+1}}_{f(x)} = 0.$$

and

$$g(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-x)^k - \frac{\lambda}{(n+1)!}(x-x)^{n+1} = f(x) - f(x) = 0.$$



By Rolle's theorem, there exists c in between $\bar{x}$ and x such that $g'(c) = 0$. Taking the derivative of g (keeping in mind that x is fixed and the independent variable is t) and using the product rule for derivatives, we have

$g'(c) = 0$

$$\begin{aligned} g'(c) &= -f'(c) + \sum_{k=1}^{n} \left(-\frac{f^{(k+1)}(c)}{k!}(x-c)^k + \frac{f^{(k)}(c)}{(k-1)!}(x-c)^{k-1} \right) + \frac{\lambda}{n!}(x-c)^n \\ &= \frac{\lambda}{n!}(x-c)^n - \frac{1}{n!}f^{(n+1)}(c)(x-c)^n \\ &= 0. \end{aligned}$$

$$\frac{k(x-c)^{k-1}}{k!} \qquad \frac{k}{k!} = \frac{1}{(k-1)!}$$

This implies $\lambda = f^{(n+1)}(c)$. The proof is now complete. □

The polynomial $P_n(x)$ given in the theorem is called the *n-th Taylor polynomial of f at* $\bar{x}$.

Remark 4.5.2 The conclusion of Taylor's theorem still holds true if $x = \bar{x}$. In this case, $c = x = \bar{x}$.

Example 4.5.1 We will use Taylor's theorem to estimate the error in approximating the function $f(x) = \sin x$ with it 3rd Taylor polynomial at $\bar{x} = 0$ on the interval $[-\pi/2, \pi/2]$. Since $f'(x) = \cos x$, $f''(x) = -\sin x$ and $f'''(x) = -\cos x$, a direct calculation shows that

$$P_3(x) = x - \frac{x^3}{3!}.$$

More over, for any $c \in \mathbb{R}$ we have $|f^{(4)}(c)| = |\sin c| \leq 1$. Therefore, for $\underbrace{x \in [-\pi/2, \pi/2]}_{|x| \leq \frac{\pi}{2}}$ we get (for some c between x and 0),

$$|\sin x - P_3(x)| = \frac{|f^{(4)}(c)|}{4!}|x| \leq \frac{\pi/2}{4!} \leq 0.066.$$

Theorem 4.5.3 Let n be an even positive integer. Suppose $f^{(n)}$ exists and continuous on (a,b). Let $\bar{x} \in (a,b)$ satisfy

$$f'(\bar{x}) = \ldots = f^{(n-1)}(\bar{x}) = 0 \text{ and } f^{(n)}(\bar{x}) \neq 0.$$

The following hold:

(a.) $f^{(n)}(\bar{x}) > 0$ iff f has local min at $\bar{x}$,

(b.) $f^{(n)}(\bar{x}) < 0$ iff f has local max at $\bar{x}$,

(a) $f^{(n)}(\bar{x}) > 0$ if and only if f has a local minimum at $\bar{x}$.
(b) $f^{(n)}(\bar{x}) < 0$ if and only if f has a local maximum at $\bar{x}$.

Proof: We will prove (a). Suppose $f^{(n)}(\bar{x}) > 0$. Since $f^{(n)}(\bar{x}) > 0$ and $f^{(n)}$ is continuous at $\bar{x}$, there exists $\delta > 0$ such that

$$f^{(n)}(t) > 0 \text{ for all } t \in B(\bar{x};\delta) \subset (a,b).$$

Fix any $x \in B(\bar{x};\delta)$. By Taylor's theorem and the given assumption, there exists c in between $\bar{x}$ and x such that

$$f(x) = f(\bar{x}) + \frac{f^{(n)}(c)}{n!}(x-\bar{x})^n.$$

Since n is even and $c \in B(\bar{x};\delta)$, we have $f(x) \geq f(\bar{x})$. Thus, f has a local minimum at $\bar{x}$.

Now, for the converse, suppose that f has a local minimum at $\bar{x}$. Then there exists $\delta > 0$ such that

$$f(x) \geq f(\bar{x}) \text{ for all } x \in B(\bar{x};\delta) \subset (a,b).$$

Fix a sequence $\{x_k\}$ in (a,b) that converges to $\bar{x}$ with $x_k \neq \bar{x}$ for every k. By Taylor's theorem, there exists a sequence $\{c_k\}$, with c_k between x_k and $\bar{x}$ for each k, such that

$$f(x_k) = f(\bar{x}) + \frac{f^{(n)}(c_k)}{n!}(x_k-\bar{x})^n.$$

Since $x_k \in B(\bar{x};\delta)$ for sufficiently large k, we have

$$f(x_k) \geq f(\bar{x})$$

for such k. It follows that

$$f(x_k) - f(\bar{x}) = \frac{f^{(n)}(c_k)}{n!}(x_k-\bar{x})^n \geq 0.$$

This implies $f^{(n)}(c_k) \geq 0$ for such k. Since $\{c_k\}$ converges to $\bar{x}$, $f^{(n)}(\bar{x}) = \lim_{k\to\infty} f^{(n)}(c_k) \geq 0$.

The proof of (b) is similar. $\square$

■ **Example 4.5.2** Consider the function $f(x) = x^2\cos x$ defined on $\mathbb{R}$. Then $f'(x) = 2x\cos x - x^2\sin x$ and $f''(x) = 2\cos x - 4x\sin x - x^2\cos x$. Then $f(0) = f'(0) = 0$ and $f''(0) = 2 > 0$. It follows from the previous theorem that f has a local minimum at 0. Notice, by the way, that since $f(0) = 0$ and $f(\pi) < 0$, 0 is not a global minimum.

■ **Example 4.5.3** Consider the function $f(x) = -x^6 + 2x^5 + x^4 - 4x^3 + x^2 + 2x - 3$ defined on $\mathbb{R}$. A direct calculations shows $f'(1) = f''(1) = f'''(1) = f^{(4)}(1) = 0$ and $f^{(5)}(1) < 0$. It follows from the previous theorem that f has a local maximum at 1.