LECTURE 22: MULTIVARIATE TAYLOR EXPANSIONS

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$n = 2$ expansion about $(x_0, y_0) = P_0$

$$f(x,y) = \underbrace{f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0)}_{L_f^{P_0}(x,y)} + \underbrace{\tfrac{1}{2} f_{xx}(P_0)(x-x_0)^2 + f_{xy}(P_0)(x-x_0)(y-y_0) + \tfrac{1}{2} f_{yy}(P_0)(y-y_0)^2}_{} + \cdots$$

$L_f^{P_0}(x,y)$ ← Linearization of $f(x,y)$ at $P_0 = (x_0, y_0)$

$$L_f^{P_0}(x,y) = f(P_0) + \nabla f(P_0) \cdot \langle x - x_0, y - y_0 \rangle$$

Hessian, well, quadratic form $Q(x,y)$. This governs variation of f at critical point. ($\nabla f(P_0) = 0$ at crit. pt.)

$n = 3$ expansion about $P_0 = (x_0, y_0, z_0)$

$$f(x,y,z) = \underbrace{f(P_0) + \nabla f(P_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle}_{} + [x-x_0, y-y_0, z-z_0] \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} + \cdots$$

$$L_f^{P_0}(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0).$$

this is the Hessian matrix, its eigenvalues tell us about the nature of a critical point

5.2 multivariate taylor series

We begin this section with a brief overview of single-variate power series. The results presented are important as we often use the single-variable results paired with a substitution to generate interesting multivariate series.

5.2.1 taylor's polynomial for one-variable

If $f : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is analytic at $x_o \in U$ then we can write

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{1}{2}f''(x_o)(x - x_o)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

We could write this in terms of the operator $D = \frac{d}{dt}$ and the evaluation of $t = x_o$

$$f(x) = \left[\sum_{n=0}^{\infty} \frac{1}{n!}(x - t)^n D^n f(t)\right]_{t=x_o}$$

I remind the reader that a function is called **entire** if it is analytic on all of $\mathbb{R}$, for example $e^x, \cos(x)$ and $\sin(x)$ are all entire. In particular, you should know that:

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Since $e^x = \cosh(x) + \sinh(x)$ it also follows that

$$\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$$

The geometric series is often useful, for $a, r \in \mathbb{R}$ with $|r| < 1$ it is known

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

This generates a whole host of examples, for instance:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots$$

$$\frac{x^3}{1-2x} = x^3(1+2x+(2x)^2+\cdots) = x^3+2x^4+4x^5+\cdots$$

Moreover, the term-by-term integration and differentiation theorems yield additional results in conjuction with the geometric series:

$$\tan^{-1}(x) = \int \frac{dx}{1+x^2} = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$

$$\ln(1-x) = \int \frac{d}{dx}\ln(1-x)dx = \int \frac{-1}{1-x}dx = -\int \sum_{n=0}^{\infty} x^n dx = \sum_{n=0}^{\infty}\frac{-1}{n+1}x^{n+1}$$

Of course, these are just the basic building blocks. We also can twist things and make the student use algebra,

$$e^{x+2} = e^x e^2 = e^2(1+x+\frac{1}{2}x^2+\cdots)$$

or trigonmetric identities,

$$\sin(x) = \sin(x-2+2) = \sin(x-2)\cos(2)+\cos(x-2)\sin(2)$$

$$\Rightarrow \quad \sin(x) = \cos(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x-2)^{2n+1} + \sin(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x-2)^{2n}.$$

Feel free to peruse my most recent calculus II materials to see a host of similarly sneaky calculations.

5.2.2 taylor's multinomial for two-variables

Suppose we wish to find the taylor polynomial centered at $(0,0)$ for $f(x,y) = e^x\sin(y)$. It is as simple as this:

$$f(x,y) = \left(1+x+\frac{1}{2}x^2+\cdots\right)\left(y-\frac{1}{6}y^3+\cdots\right) = y+xy+\frac{1}{2}x^2y-\frac{1}{6}y^3+\cdots$$

the resulting expression is called a multinomial since it is a polynomial in multiple variables. If all functions $f(x,y)$ could be written as $f(x,y) = F(x)G(y)$ then multiplication of series known from calculus II would often suffice. However, many functions do not possess this very special form. For example, how should we expand $f(x,y) = \cos(xy)$ about $(0,0)$?. We need to derive the two-dimensional Taylor's theorem[3].

In previous chapters we have discussed the best linear approximation for a function of several variables. The next step is the best quadratic approximation. In particular, we seek to find formulas to fix the constants $c_o, c_1, c_2, c_{11}, c_{12}, c_{22}$ as given below:

$$\boxed{f(x,y) \approx c_o + c_1(x-x_o) + c_2(y-y_o) + c_{11}(x-x_o)^2 + c_{12}(x-x_o)(y-y_o) + c_{22}(y-y_o)^2.}$$

[3] A more careful proof will be found in most advanced calculus texts, it turns out the multivariate expansion follow from differentiating $g = f\circ\vec{r}$ where $\vec{r}:\mathbb{R}\to\mathbb{R}^2$ has $\vec{r}(t) = \langle x_o+at, y_o+bt\rangle$. The single-variable Taylor theorem applies and we therefore generalize the remainder estimation theorems to higher dimensions. I will not dig deeper into questions about the remainder for multivariate taylor expansions in these notes. Intuitively, we have one assumption: higher order terms are small near the center of the series.

The expression above is a quadratic polynomial in x, y centered at (x_o, y_o). Observe that it is already clear that $f(x_o, y_o) = c_o$. Take partial derivatives in x and y,

$$f_x(x,y) \approx c_1 + 2c_{11}(x - x_o) + c_{12}(y - y_o) \qquad f_y(x,y) \approx c_2 + c_{12}(x - x_o) + 2c_{22}(y - y_o).$$

Therefore, it is clear that: $f_x(x_o, y_o) = c_1$ and $f_y(x_o, y_o) = c_2$. Differentiating once more,

$$f_{xx}(x,y) \approx 2c_{11} \qquad f_{xy}(x,y) \approx c_{12} \qquad f_{yy}(x,y) \approx 2c_{22}.$$

Therefore, $f_{xx}(x_o, y_o) = 2c_{11}$, $f_{xy}(x_o, y_o) = c_{12}$ and $f_{yy}(x_o, y_o) = 2c_{22}$. It follows that we can construct the best quadratic approximation near (x_o, y_o) by the formula below: let $\vec{p}_o = (x_o, y_o)$

$$\boxed{f(x,y) \approx f(x_o, y_o) + L(x - x_o, y - y_o) + Q(x - x_o, y - y_o)}$$

Where, I denoted $L(x - x_o, y - y_o) = f_x(\vec{p}_o)(x - x_o) + f_y(\vec{p}_o)(y - y_o)$ and

$$Q(x - x_o, y - y_o) = \frac{1}{2} f_{xx}(\vec{p}_o)(x - x_o)^2 + f_{xy}(\vec{p}_o)(x - x_o)(y - y_o) + \frac{1}{2} f_{yy}(\vec{p}_o)(y - y_o)^2.$$

Notice that $f(\vec{p}_o) + L(x - x_o, y - y_o)$ gives the first-order approximation of f, it is the linearization of f at $\vec{p}_o$. We can also write the expansion as

$$\boxed{f(x_o + h, y_o + k) \approx f(\vec{p}_o) + f_x(\vec{p}_o)h + f_y(\vec{p}_o)k + \frac{1}{2} f_{xx}(\vec{p}_o)h^2 + f_{xy}(\vec{p}_o)hk + \frac{1}{2} f_{yy}(\vec{p}_o)k^2.}$$

Example 5.2.1. *Suppose $f(x,y) = \sqrt{1 + x + y}$. Differentiating yields:*

$$f_x(x,y) = f_y(x,y) = \frac{1}{2}(1 + x + y)^{-1/2}.$$

Differentiate once more,

$$f_{xx}(x,y) = f_{yy}(x,y) = f_{xy}(x,y) = \frac{-1}{4}(1 + x + y)^{-3/2}.$$

Observe that $f(0,0) = 1$, $f_x(0,0) = f_y(0,0) = \frac{1}{2}$ and $f_{xx}(0,0) = f_{xy}(0,0) = f_{yy}(0,0) = \frac{-1}{4}$. Therefore,

$$\sqrt{1 + x + y} \approx 1 + \frac{1}{2}(x + y) - \frac{1}{8}(x^2 + 2xy + y^2)$$

As an application, let's calculate $\sqrt{1.11}$. Notice $1.11 = 1 + 0.1 + 0.01$ so apply the formula with $x = 0.1$ and $y = 0.01$,

$$\begin{aligned}
\sqrt{1 + 0.1 + 0.01} &\approx 1 + \frac{1}{2}(0.1 + 0.01) - \frac{1}{8}((0.1)^2 + 2(0.1)(0.01) + (0.01)^2) \\
&\approx 1 + (0.5)(0.11) - (0.125)(0.01 + 0.002 + 0.0001) \\
&\approx 1 + (0.5)(0.11) - (0.125)(0.0121) \\
&\approx 1 + 0.055 - 0.0015125 \\
&\approx 1 + 0.055 - 0.0015125 \\
&\approx 1.0534875.
\end{aligned}$$

In contrast, my Casio fx-115 ES claims $\sqrt{1.11} = 1.053565375$. If we trust my calculator then we have correctly calculated the four correct digits for $\sqrt{1.11}$. Not too shabby for our trouble.

Computation of third, fourth or higher order terms reveals the multivariate taylor expansion below. We denote $h = h_1$ and $k = h_2$,

$$f(x_o + h, y_o + k) = \sum_{n=0}^{\infty}\sum_{i_1=0}^{2}\sum_{i_2=0}^{2}\cdots\sum_{i_n=0}^{2}\frac{1}{n!}\frac{\partial^{(n)}f(x_o, y_o)}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_n}}h_{i_1}h_{i_2}\cdots h_{i_n}$$

Example 5.2.2. *Expand $f(x, y) = \cos(xy)$ about $(0, 0)$. We calculate derivatives,*

$$f_x = -y\sin(xy) \qquad f_y = -x\sin(xy)$$

$$f_{xx} = -y^2\cos(xy) \qquad f_{xy} = -\sin(xy) - xy\cos(xy) \qquad f_{yy} = -x^2\cos(xy)$$

$$f_{xxx} = y^3\sin(xy) \qquad f_{xxy} = -y\cos(xy) - y\cos(xy) + xy^2\sin(xy)$$

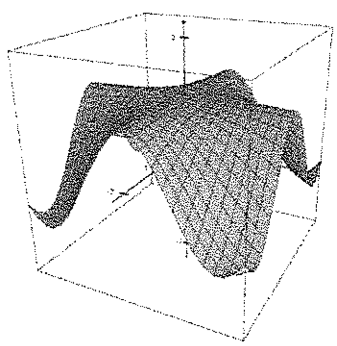
$$f_{xyy} = -x\cos(xy) - x\cos(xy) + x^2y\sin(xy) \qquad f_{yyy} = x^3\sin(xy)$$

Next, evaluate at $x = 0$ and $y = 0$ to find $f(x, y) = 1 + \cdots$ to third order in x, y about $(0, 0)$. We can understand why these derivatives are all zero by approaching the expansion a different route: simply expand cosine directly in the variable (xy),

$$f(x, y) = 1 - \frac{1}{2}(xy)^2 + \frac{1}{4!}(xy)^4 + \cdots = 1 - \frac{1}{2}x^2y^2 + \frac{1}{4!}x^4y^4 + \cdots .$$

Apparently the given function only has nontrivial derivatives at $(0, 0)$ at orders $0, 4, 8, \ldots$. We can deduce that $f_{xxxxy}(0, 0) = 0$ without further calculation.

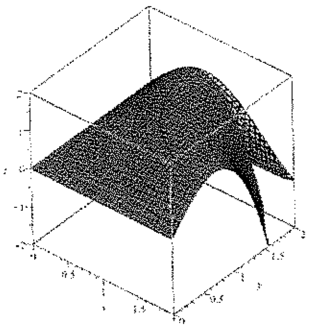
This is actually a very interesting function, I think it defies our analysis in the later portion of this chapter. The second order part of the expansion reveals nothing about the nature of the critical point $(0, 0)$. Of course, any student of trigonometry should recognize that $f(0, 0) = 1$ is likely a local maximum, it's certainly not a local minimum. The graph reveals that $f(0, 0)$ is a local maxium for f restricted to certain rays from the origin whereas it is constant on several special directions (the coordinate axes).

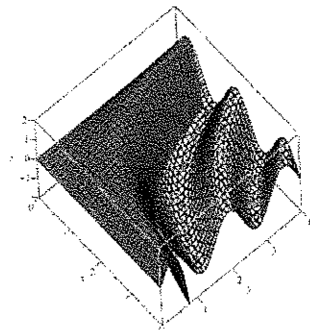


Example 5.2.3. *Suppose $f(x, y) = \sin(xy)$. Once more I'll use the substitution trick. Let $u = xy$ hence $f(x, y) = \sin(u) = u - \frac{1}{6}u^3 + \cdots$ and to quadratic 6-th order we find*

$$f(x, y) = xy - \frac{1}{6}x^3y^3 + \cdots .$$

It is interesting to compare the graph $z = f(x, y)$ and $z = xy - \frac{1}{6}x^3y^3$, note how closely they correspond near the origin: the red graph is the approximating surface $z = xy - \frac{1}{6}x^3y^3$ and the transparent wire-frame is the actual function $z = \sin(xy)$. Roughly, they are within 0.1 units a distance of 1 from the origin. You can see in the right picture as we zoom away they difference between the function and the approximation is appreciable.





Example 5.2.4. *Suppose $f(x,y) = \sin(\ \sqrt{x^2+y^2}\ )$ then in polar coordinates $f(r,\theta) = \sin(r)$. In this case the natural expansion to use is $f(x,y) = r - \frac{1}{6}r^3 + \frac{1}{120}r^5 + \cdots$ which is not technically a multivariate power series in x, y. In fact, $\sqrt{x^2+y^2}$ is not even differentiable at $(0,0)$.*

Example 5.2.5. *Suppose $f(x,y) = \sin(\ x^2+y^2\ )$ then in polar coordinates $f(r,\theta) = \sin(r^2)$. In this case the natural expansion to use is $f(x,y) = r^2 - \frac{1}{6}r^6 + \frac{1}{120}r^{10} + \cdots$ which is easily rewritten as a multivariate power series since $r^2 = x^2 + y^2$. You use the series to observe that the first, third, fourth, fifth, seventh, eighth and ninth derivatives of f at $(0,0)$ are zero.*

5.2.3 taylor's multinomial for many-variables

Suppose $f : dom(f) \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function of n-variables. It turns out that the Taylor series centered at $\vec{a} = (a_1, a_2, \dots, a_n)$ has the form:

$$\boxed{f(\vec{a}+\vec{h}) = \sum_{k=0}^{\infty}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}\frac{1}{k!}(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}f)(\vec{a})\ h_{i_1}h_{i_2}\cdots h_{i_k}.}$$

Naturally, we sometimes prefer to write the series expansion about $\vec{a}$ as an expresssion in $\vec{x} = \vec{a}+\vec{h}$. With this substitution we have $\vec{h} = \vec{x} - \vec{a}$ and $h_{i_j} = (x-a)_{i_j} = x_{i_j} - a_{i_j}$ thus

$$\boxed{f(x) = \sum_{k=0}^{\infty}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}\frac{1}{k!}(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}f)(\vec{a})\ (x_{i_1}-a_{i_1})(x_{i_2}-a_{i_2})\cdots(x_{i_k}-a_{i_k}).}$$

Proof of these claims is found in advanced calculus. Let me illustrate how these formulas work for $n = 3$.

Example 5.2.6. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ let's unravel the Taylor series centered at $(0,0,0)$ from the general formula boxed above. Utilize the notation $x = x_1, y = x_2$ and $z = x_3$ in this example.*

$$f(\vec{x}) = \sum_{k=0}^{\infty}\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\cdots\sum_{i_k=1}^{3}\frac{1}{k!}(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}f)(0)\ x_{i_1}x_{i_2}\cdots x_{i_k}.$$

The terms to order 2 are as follows:

$$\begin{aligned} f(\vec{x}) &= f(0) + f_x(0)x + f_y(0)y + f_z(0)z \\ &+\tfrac{1}{2}\Big(f_{xx}(0)x^2 + f_{yy}(0)y^2 + f_{zz}(0)z^2 + \\ &\quad + f_{xy}(0)xy + f_{xz}(0)xz + f_{yz}(0)yz + f_{yx}(0)yx + f_{zx}(0)zx + f_{zy}(0)zy \Big) + \cdots \end{aligned}$$

Partial derivatives commute for smooth functions hence,

$$f(\vec{x}) = \underbrace{f(0) + f_x(0)x + f_y(0)y + f_z(0)z}_{linearization} + \\ + \underbrace{\frac{1}{2}\Big(f_{xx}(0)x^2 + f_{yy}(0)y^2 + f_{zz}(0)z^2 + 2f_{xy}(0)xy + 2f_{xz}(0)xz + 2f_{yz}(0)yz \Big)}_{quadratic\ form\ Q(x,y,z)} + \cdots$$

Identify that $f(0) + f_x(0)x + f_y(0)y + f_z(0)z$ is the linearization of f at the origin and the quadratic terms are simply the analogue of $Q(x,y) = f_xx(0)x^2 + 2f_{xy}(0)xy + f_{yy}(0)y^2$ for $n = 3$.

In the $n = 2$ case the graph $z = f(x,y)$ is relatively easy to visualize. Intuitively, the lineariization gives a plane which resembles the graph and then the linearization plus the quadratic form give some quadratic surface which better models the graph $z = f(x,y)$ neat the point of the expansion. Something similar is true in $n = 3$ however visualization is hard since the graph $w = f(x,y,z)$ is a four-dimensional picture.

Example 5.2.7. *Suppose $f(x,y,z) = e^{xyz}$. Find a quadratic approximation to f near $(0,1,2)$. Observe:*

$$f_x = yze^{xyz} \qquad f_y = xze^{xyz} \qquad f_z = xye^{xyz}$$

$$f_{xx} = (yz)^2e^{xyz} \qquad f_{yy} = (xz)^2e^{xyz} \qquad f_{zz} = (xy)^2e^{xyz}$$

$$f_{xy} = ze^{xyz} + xyz^2e^{xyz} \qquad f_{yz} = xe^{xyz} + x^2yze^{xyz} \qquad f_{xz} = ye^{xyz} + xy^2ze^{xyz}$$

Evaluating at $x = 0, y = 1$ and $z = 2$,

$$f_x(0,1,2) = 2 \qquad f_y(0,1,2) = 0 \qquad f_z(0,1,2) = 0$$

$$f_{xx}(0,1,2) = 4 \qquad f_{yy}(0,1,2) = 0 \qquad f_{zz}(0,1,2) = 0$$

$$f_{xy}(0,1,2) = 2 \qquad f_{yz}(0,1,2) = 0 \qquad f_{xz}(0,1,2) = 1$$

x
(y−1)
(z−2)
(0, 1, 2)

Hence, as $f(0,1,2) = e^0 = 1$ we find

$$f(x,y,z) = 1 + 2x + 2x^2 + 2x(y-1) + 2x(z-2) + \cdots$$

Another way to calculate this expansion is to make use of the adding zero trick, $e^\theta = 1+\theta+\frac{1}{2}\theta^2+\frac{1}{3!}\theta^3+\cdots$

$$f(x,y,z) = e^{x(y-1+1)(z-2+2)} = 1 + x(y-1+1)(z-2+2) + \frac{1}{2}\big[x(y-1+1)(z-2+2)\big]^2 + \cdots$$

Keeping only terms with two or less of x, $(y-1)$ and $(z-2)$ variables,

$$f(x,y,z) = 1 + 2x + x(y-1)(2) + x(1)(z-2) + \frac{1}{2}x^2(1)^2(2)^2 + \cdots$$

Which simplifies once more to $f(x,y,z) = 1 + 2x + 2x(y-1) + x(z-2) + 2x^2 + \cdots$.

Example 5.2.8. *Suppose $f(x,y,z) = \frac{1}{1-z^2}e^{x^2}\cos(y^3)$. Find the multivariate series expansion to quadratic order about the origin. In this case we can just multiply expansions known from calculus II, no need to do partial derivatives!*

$$f(x,y,z) = \Big(1 - z^2 + z^4 + \ldots\Big)\Big(1 + x^2 + \frac{1}{2}x^4 + \cdots\Big)\Big(1 - \frac{1}{2}y^6 + \cdots\Big)$$
$$= 1 + x^2 - z^2 + \cdots.$$