LECTURE 23

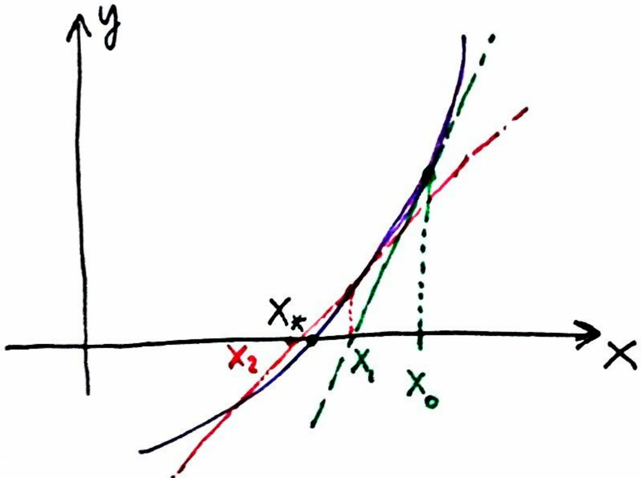
NEWTON'S METHOD & CONTRACTION MAPPINGS

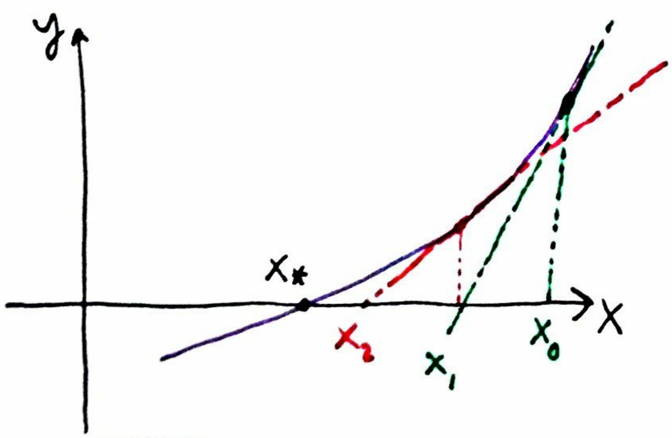
- Based on pgs 160 – 164 of C.H. Edwards, JR. ADVANCED CALCULUS OF SEVERAL VARIABLES

$\boxed{\text{PROBLEM: Solve } f(x) = 0.}$

- Simplifying assumptions:
  1. $f: \mathbb{R} \to \mathbb{R}$ is differentiable with continuous f'
  2. $f'(x) \neq 0 \;\; \forall x \in [a,b]$ and $f(x)$ changes sign on $[a,b]$

Indicates $\exists!\, x_* \in [a,b]$ for which $f(x_*) = 0$





NEWTON'S METHOD

- GUESS $x_0 \in [a,b]$ as potential zero. Check $f(x_0)$ if $f(x_0) \neq 0$ then study tangent line based at x_0,

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Observe

$$0 = f(x_0) + f'(x_0)(x - x_0)$$

Has sol^n

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

thus set

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

as next guess. Check $f(x_1) \overset{?}{=} 0$
If not then,

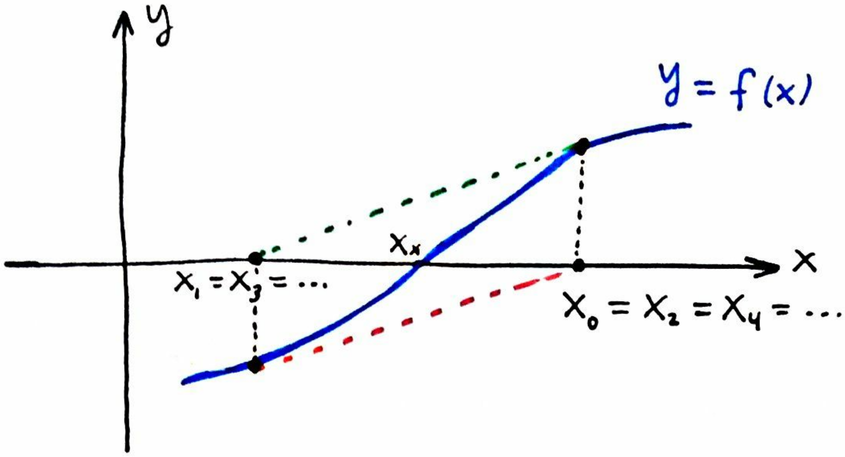
$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

and... continue, setting

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until either $f(x_{n+1}) = 0$ or it's close "enough"...

TROUBLESOME EXAMPLE FOR "NAIVE" NEWTON'S METHOD



MODIFIED MAX-SLOPE NEWTON

Let $M = \max|f'(x)|$ if $f'(x) > 0$ for $x \in [a,b]$.
Let $M = -\max|f'(x)|$ if $f'(x) < 0$ for $x \in [a,b]$.
Then define the sequence $\{x_n\}$ recursively by

$$x_{n+1} = x_n - \frac{f(x_n)}{M}$$

where $x_0 \in [a,b]$ is some initial guess for solving $f(x) = 0$.
Once again we've assumed $f'(x) \neq 0 \ \forall x \in [a,b]$ and f' is continuous, and $f(a)f(b) < 0$.

Thm/ The modified max-slope Newton's Method converges to the unique solution to $f(x) = 0$; $x_n \to x_*$ with $f(x_*) = 0$. Moreover, the error in the n^{th} iterate is bounded as follows,

$$|x_n - x_*| \leq \frac{|f(x_0)|}{m}\left(1 - \frac{m}{M}\right)^n$$

where $m < f'(x) < M$

and $f(a) < 0 < f(b)$

Proof: $\varphi(x) = x - \frac{f(x)}{M}$ is a contraction mapping on $[a,b]$ — discussed next

hence $x_{n+1} = \varphi(x_n)$ recursively defines sequence for which $x_n \to x_*$ with $\varphi(x_*) = x_*$. However,

$$\varphi(x_*) = x_* = x_* - \frac{f(x_*)}{M} \Rightarrow \underline{f(x_*) = 0}.$$

Remark: this is not the complete proof, it's a summary sketch. Also, the Thm holds with suitable modification when $f'(x) < 0$ etc...

Def^{n}/ $\varphi : [a,b] \to [a,b]$ is a contraction mapping with contraction constant $k < 1$ if $|\varphi(x) - \varphi(y)| \leq k|x-y| \ \forall x,y \in [a,b]$

Th^{m}/ (contraction mapping theorem) (Edward's Th^{m} 1.1 on pg. 162)

Let $\varphi : [a,b] \to [a,b]$ be contraction mapping with contraction constant $k < 1$. Then φ has a unique fixed point $x_* \in [a,b]$; meaning $\exists!\, x_* \in [a,b]$ for which $\varphi(x_*) = x_*$. Furthermore, given $x_0 \in [a,b]$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined recursively by $x_{n+1} = \varphi(x_n)$ converges to x_*. In fact,

$$|x_n - x_*| \leq \frac{k^n |x_0 - x_1|}{1-k}$$

Proof: observe $|x_{n+1} - x_n| = |\varphi(x_n) - \varphi(x_{n-1})| \leq k|x_n - x_{n-1}|$ from which we can prove by induction that $|x_{n+1} - x_n| \leq k^n |x_1 - x_0|$. ★
Therefore, for $0 < n < m$ we calculate,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} + \cdots + x_{m-1} - x_m| \quad \Delta\text{-inequality} \\
&\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \quad ★ \\
&\leq k^n|x_1 - x_0| + k^{n+1}|x_1 - x_0| + \cdots + k^{m-1}|x_1 - x_0| \\
&\leq (k^n + k^n k + \cdots + k^n k^{m-n-1})|x_1 - x_0| \\
&\leq k^n(1 + k + k^2 + \cdots)|x_1 - x_0| \quad : \text{geometric series! } 1 + k + k^2 + \cdots = \frac{1}{1-k} \\
&\leq \frac{k^n |x_1 - x_0|}{1-k} \quad \Rightarrow \{x_n\} \text{ is CAUCHY SEQUENCE}
\end{aligned}$$

(Let $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{k^N|x_1 - x_0|}{1-k} < \varepsilon$, since $k < 1$ this is a reasonable choice, then if $N < n < m$ then $|x_n - x_m| < \varepsilon$.)

Then $\{x_n\}$ Cauchy implies $x_n \to x_* \in [a,b]$ ← closed interval. as $n \to \infty$. Moreover, $m \to \infty$

$$|x_n - x_m| \leq \frac{k^n|x_1 - x_0|}{1-k} \Rightarrow |x_n - x_*| \leq \frac{k^n|x_1 - x_0|}{1-k}.$$

If $\varphi(x_*) = x_*$ and $\varphi(x_{**}) = x_{**}$ then $|x_* - x_{**}| = |\varphi(x_*) - \varphi(x_{**})| \leq k|x_* - x_{**}|$ thus $|x_* - x_{**}| = 0$ and so $x_* = x_{**}$. The fixed pt x_* is unique. //

Th$^{\underline{m}}$ (1.2 Edwards, pg. 164)

Let $f : [a,b] \longrightarrow \mathbb{R}$ be differentiable function with $f(a) < 0 < f(b)$ and $0 < m < f'(x) \leq M$ for $x \in [a,b]$. Given $x_0 \in [a,b]$ the sequence defined recursively by

$$x_{n+1} = x_n - \frac{f(x_n)}{M}$$

converges to the unique root $x_* \in [a,b]$ for which $f(x_*) = 0$. Moreover, $|x_n - x_*| \leq \frac{|f(x_0)|}{m}\left(1 - \frac{m}{M}\right)^n$ for each $n \in \mathbb{N}$.

Proof: Let $\varphi : [a,b] \longrightarrow \mathbb{R}$ be defined by $\varphi(x) = x - \frac{f(x)}{M}$.

Observe $\varphi'(x) = 1 - \frac{f'(x)}{M}$ and as $\underbrace{0 < m < f'(x) \leq M}$

$$0 < \frac{m}{M} < \frac{f'(x)}{M} \leq \frac{M}{M} = 1$$

$$\Rightarrow 0 \leq 1 - \frac{f'(x)}{M} = \varphi'(x)$$

Thus $0 \leq \varphi'(x) = 1 - \frac{f'(x)}{m} < 1 - \frac{m}{M} = k < 1$.

Therefore, φ is an increasing function and we may observe,

$$a < a - \frac{f(a)}{M} = \varphi(a) \leq \varphi(x) \leq \varphi(b) = b - \frac{f(b)}{M} < b \quad (\star)$$

for all $x \in [a,b]$. Thus $\varphi([a,b]) \subseteq [a,b]$. Suppose $x,y \in [a,b]$ and $x \neq y$ then by the Mean Value Th$^{\underline{m}}$ $\exists c \in [a,b]$ for which

$$\frac{\varphi(x) - \varphi(y)}{x - y} = \varphi'(c) < k \Rightarrow |\varphi(x) - \varphi(y)| \leq k|x-y|$$

thus the bound on $\varphi'(x)$ by k suffices with $(\star)$ to show φ is contraction mapping on $[a,b]$ with $k = 1 - \frac{m}{M}$. If we note $\varphi(x_0) = x_0 - \frac{f(x_0)}{m}$ then $x_1 = x_0 - \frac{f(x_0)}{M}$ and so $|x_1 - x_0| = \frac{|f(x_0)|}{M}$ thus by contraction mapping Th$^{\underline{m}}$ $\exists!\, x_* \in [a,b]$ s.t. $x_n \longrightarrow x_*$ where $\varphi(x_n) = x_{n+1} \longleftrightarrow x_{n+1} = x_n - \frac{f(x_n)}{M}$.

Furthermore,

$$|x_n - x_*| \leq \frac{k^n |x_1 - x_0|}{1-k} = \frac{|f(x_0)|}{M}\left(1 - \frac{m}{M}\right)^n \frac{1}{1 - (1 - \frac{m}{M})}$$

$$= \frac{|f(x_0)|}{m}\left(1 - \frac{m}{M}\right)^n. //$$

A LOOK AHEAD IN EDWARDS

The argument given in this Lecture for Newton's Method can be adapted to solve the problem of calculating the inverse function to given function... well, to be precise, given $f'(x) \neq 0$, can choose nbhd close to x for which f suitably restricted has inverse $f^{-1}(y)$.

Algebraically we're trying to solve for x:

$$f(x) = y \implies x = f^{-1}(y)$$

Well, suppose $f(a) = b$ and x is close to a $(f'(a) \neq 0)$ then approximately

$$y - b = f(x) - f(a) \approx f'(a)(x-a)$$

$$\Rightarrow x \cong a - \frac{f(a)-y}{f'(a)}$$

Setting $x_0 = a$ we suspect, defining

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)-y}{f'(x_n)}}$$

(1.3 Edwards, pg. 166)

Thm/ $f: \mathbb{R} \to \mathbb{R}$ a C' function with $f(a) = b$ and $f'(a) \neq 0$. Then $\exists$ nbhds $U = [a-\delta, a+\delta]$ of a and $V = [b-\varepsilon, b+\varepsilon]$ of b such that given $y_* \in V$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 = a$ & $x_{n+1} = x_n - \frac{f(x_n) - y_*}{f'(a)}$

converges to a unique point $x_* \in U$ such that $f(x_*) = y_*$.

Can define local inverse of f to be g where $g(y_*) = x_*$. The function g can be seen as limit of sequence of functions

$$g_0(y) = a \quad \& \quad g_{n+1}(y) = g_n(y) - \frac{f(g_n(y)) - y}{f'(a)}$$

Example 1 from pg. 166 Edwards

$$f(x) = x^2 - 1$$

$$a = 1,\ b = 0 \quad \text{as} \quad f(1) = 1^2 - 1 = 0.$$

$$g_0(y) \equiv 1$$

$$g_1(y) = 1 - \frac{f(g_0(y)) - y}{f'(1)} = 1 - \frac{[1^2 - 1] - y}{2} = 1 + \frac{y}{2}$$

$$\left.\begin{aligned} g_2(y) &= g_1(y) - \tfrac{1}{2}\left[f(g_1(y)) - y\right] \\ &= 1 + \tfrac{y}{2} - \tfrac{1}{2}\left[\left(1 + \tfrac{y}{2}\right)^2 - 1 - y\right] \\ &= 1 + \tfrac{y}{2} + \tfrac{1}{2}\left[\cancel{1} + \cancel{y} + \tfrac{y^2}{4} \cancel{-1} - \cancel{y}\right] \\ &= 1 + \tfrac{y}{2} + \tfrac{1}{2}\left[\tfrac{y^2}{4}\right] \\ &= 1 + \tfrac{y}{2} - \tfrac{1}{8} y^2 \end{aligned}\right\} \text{sorry for mess}$$

$$g_3(y) = 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^4}{16} - \frac{y^4}{128}$$

This is neat since if we solve $x^2 - 1 = y$ for $x > 0$ we obtain $x = \sqrt{1 + y}$ and

$$(1+y)^\alpha = 1 + \alpha y + \tfrac{1}{2}\alpha(\alpha - 1) y^2 + \tfrac{1}{3!}\alpha(\alpha-1)(\alpha-2) y^3 + \cdots$$

$$= 1 + \tfrac{1}{2} y - \tfrac{1}{8} y^2 + \tfrac{1}{3 \cdot 2}\, \tfrac{1}{2}\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right) y^3 + \tfrac{1}{4!}\, \tfrac{1}{2}\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)\left(\tfrac{-5}{2}\right) y^4 + \cdots$$

$$= 1 + \tfrac{1}{2} y - \tfrac{1}{8} y^2 + \tfrac{1}{16} y^3 - \tfrac{5}{128} y^4 + \cdots$$