LECTURE 24: CLOSED SET TEST

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5.4 closed set method

The analog of the extreme value theorem of first semester calculus is given below.

Theorem 5.4.1.

If D is a closed and bounded subset of $\mathbb{R}^2$ then a continuous function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ attains a global maximum and global minimum somewhere in D.

To say D is closed means that it has edges which are not fuzzy in our usual contexts. There is a better *topological* method to describe such terms[5] but I leave that for another course. To say D is bounded simply means we can find a point $(x_o, y_o) \in D$ and $\epsilon > 0$ such that $D \subset B_\epsilon(x_o, y_o) = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{(x-x_o)^2 + (y-y_o)^2} < \epsilon\}$. In other words, D is bounded if there exists a finite open disk which properly contains D. Or, in plain-English, if you can draw a circle big enough to enclose D. These terms don't usually bother students in practice, if anything, the attempt to define them here is the most troubling part. Common examples of closed and bounded sets are: disks, rectangles, areas bounded by curves which we studied in first semester calculus, polygons regular or otherwise.

The extreme value theorem told us that the maximum and minimum values of a continuous function on a closed interval $[a,b]$ were attained somewhere in $[a,b]$. That data motivated the **closed-interval test** which said, given a continuous function on a closed interval $[a,b]$,

(i.) find any critical numbers for f in the interval

(ii.) evaluate the function at critical numbers and endpoints

(iii.) select the minimum and maximum from the values found in step **(ii.)**

The theorem that follows is the analog of the closed interval test for functions of several variables.

Theorem 5.4.2.

Suppose D is a closed and bounded subset of $\mathbb{R}^2$ and $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Extreme values of f on D may be found as follows:

(i.) find the value of f at any critical points in the interior of D

(ii.) find any extreme values for f on the boundary of D

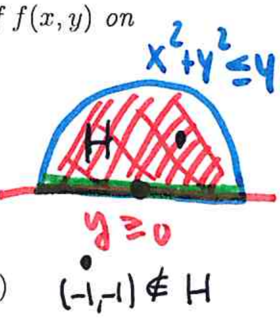
(iii.) select the minimum and maximum on D from the values found in steps **(i.)** and **(ii.)**

I leave the proof of this assertion to another course. That said, it is useful to think about the two cases[6]. As we consider closed and bounded D it follows $D = \text{int}(D) \cup \partial D$. The boundary ∂D is the edge whereas $\text{int}(D)$ is D with the edge removed. The basic idea is that we can apply the theory of local extrema to the interior; that is, use the second derivative test to classify any critical points in the interior. On the other hand, the boundary is a curve or set of curves where we might apply the method of Lagrange multipliers. However, sometimes the boundary admits a better solution in terms of a parametric formulation. We'll see that techinque in the examples to follow below.

[5] indeed, you may learn later that closed and bounded is synonomous with compact

[6] I should admit, I assume f is continuously differentiable in this discussion as to avoid certain pathological cases.

Example 5.4.3. *Let $f(x,y) = x^4 + y^4 - 4xy + 1$. Find the maximum and minimum of $f(x,y)$ on the half-disk $H = \{(x,y) \mid x^2 + y^2 \leq 4,\ y \geq 0\}$.*



Solution: *we begin by searching for local maxima and minima. Consider,*

$$\nabla f(x,y) = \langle 4x^3 - 4y, 4y^3 - 4x \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad y = x^3 \ \ \& \ \ x = y^3.$$

It follows $x^9 = x$. This is solved by factoring,

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 + 1)(x^4 - 1) = x(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$$

Hence $x = 0, 1, -1$ and we find critical points $(0,0), (1,1)$ and $(-1,-1)$. Note, $(-1,-1) \notin H$ hence we ignore it. Notice $f_{xx} = 12x^2$ and $f_{yy} = 12y^2$ and $f_{xy} = -4$ give Hessian $D = 144x^2y^2 - 16$. Hence,

critical point	D	f_{xx}	*conclusion*
$(0,0)$	-16	*no need*	$f(0,0) = 1$ *is a saddle*
$(1,1)$	128	12	$f(1,1) = -1$ *is local minimum*

Logically, we do not need the Hessian or the analysis of the table above (In include it here for curiousity alone). It suffices to calculate $f(0,0), f(1,1)$ and $f(-1,-1)$ for future comparison to extreme values on the boundary ∂H. There are two cases in the boundary:

(i.) *the diameter of the half-circle boundary is given by $y = 0$ and $-2 \leq x \leq 2$. Let $g(x) = f(x,0) = x^4 + 1$. We analyze the behaviour of g on $[-2,2]$ by the closed interval test. Notice $g'(x) = 4x^3$ hence $x = 0$ is the only critical number. Observe,*

$$g(-2) = 17, \quad g(0) = 1, \quad g(2) = 17.$$

Thus, $f(-2,0) = 17$, $f(2,0) = 17$ are two new candidates we should consider as we seek the extreme values of f on H.

(ii.) *curved part of the half-circle has parameterization $x = 2\cos t$ and $y = 2\sin t$ for $0 \leq t \leq \pi$. Let $h(t) = f(2\cos t, 2\sin t)$ which gives $h(t) = 16(\sin^4 t + \cos^4 t - \sin t \cos t)$. We find extrema of h on $[0,\pi]$ by the closed interval test. Consider,*

$$h'(t) = 16(-4\sin t \cos^3 t + 4\cos t \sin^3 t - \cos^2 t + \sin^2 t).$$

thus $h'(t) = 0$ yields:

$$-(\cos(t) - \sin(t))(\sin(t) + \cos(t))(4\sin(t)\cos(t) + 1) = 0$$

or,

$$\tan t = 1, \quad \tan t = -1, \quad \sin(2t) = -1/2.$$

We seek solutions on $[0,\pi]$. Observe, $t = \pi/4$ give $\tan(\pi/4) = 1$. Also, $t = 3\pi/4$ gives $\tan(3\pi/4) = -1$. Solutions of $\sin(2t) = -1/2$ are $2t = 7\pi/6$ and $2t = 11\pi/6$ hence $t = 7\pi/12$ and $t = 11\pi/12$. These four values of t yield points:

$$(\sqrt{2}, \sqrt{2}), \quad (-\sqrt{2}, \sqrt{2}), \quad (2\cos(7\pi/12), 2\sin(7\pi/12)), \quad (2\cos(11\pi/12), 2\sin(11\pi/12)).$$

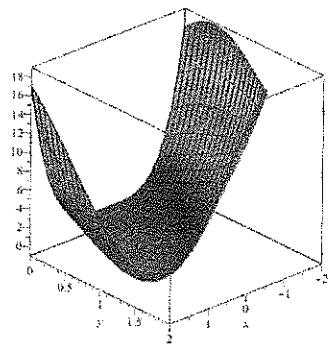
Or, approximately,

$$(1.41, 1.41), \quad (-1.41, -1.41), \quad (-0.52, 1.93), \quad (-1.93, 0.52).$$

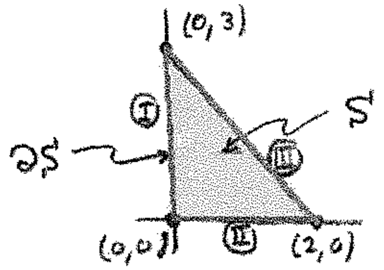
These yield (approximate) values:

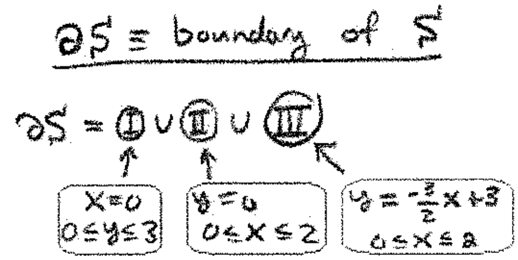
$$f(1.41, 1.41) = 0.31, \ \ f(-1.41, -1.41) = 11.6, \ \ f(-0.52, 1.93) = 19.0, \ \ f(-1.93, 0.52) = 19.0.$$

Of the nine possible extremal points, we observe the minimum value is -1 which is attained at $(1,1)$ and the maximum value is approximately 19 which is attained at $(2\cos(7\pi/12), 2\sin(7\pi/12))$ and $(2\cos(11\pi/12), 2\sin(11\pi/12))$. The graph below illustrates our analysis:



Example 5.4.4. *Consider $f(x,y) = 1 + 4x - 5y$. Find the absolute extrema of f on the set S pictured below: I have taken the step of labeling the edges for convenience of discussion.*





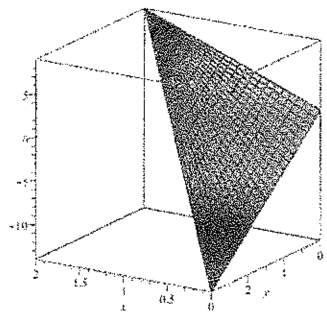
To begin note that $\nabla f(x,y) = \langle 4, -5\rangle \neq 0$ thus there is no local extrema in the interior of S. We need only consider ∂S

(I.) $x = 0$ *and* $y \in [0,3]$. *Let $g(y) = f(0,y) = 1 - 5y$. Note $g'(y) = -5 \neq 0$ hence the closed interval test need only consider $g(0) = 1$ and $g(3) = -14$. For future reference, we should remember to consider $f(0,0) = 1$ and $f(0,3) = -14$ as possible extrema on S.*

(II.) $y = 0$ *and* $x \in [0,2]$. *Let $h(x) = f(x,0) = 1 + 4x$. Note $h'(x) = 4 \neq 0$ hence the closed interval test faces no critical numbers. We consider the endpoints; $h(0) = f(0,0) = 1$ and $h(2) = f(2,0) = 9$. This shows $1 \leq f(x,0) \leq 9$ for $0 \leq x \leq 2$.*

(III.) $y = -\frac{3}{2}x + 3$ *for* $x \in [0,2]$. *Let $l(x) = f(x, -\frac{3}{2}x + 3) = -14 + \frac{23}{2}x$. Once again, $l'(x) = -\frac{23}{2} \neq 0$ hence $l(0) = -14$ and $l(2) = 9$ are possible extrema for l on $[0,2]$. Once more, we are prompted to consider $f(0,0) = -14$ and $f(2,0) = -14$ as possible extreme values for S.*

In summary, only the vertices of the triangular region appear as possible extrema and we conclude the **maximum of f on S** *is 9 which is attained at $(2,0)$ and the* **minimum of f on S** *is -14 which is attained at $(0,0)$. Geometrically, our analysis is easy to see: here I plot $z = 1 + 4x - 5y$ for $(x,y) \in S$*



The result above generalizes to any closed polygon in the plane. If we find extrema of $f(x,y) = ax+by+c$ for some a,b with $ab \neq 0$ then we need only consider vertices of the polygon. This well-known result is often taught in high-school algebra as **linear programming**. If we consider the higher-dimensional problem with linear constraints in three variables which enclose some polyhedral surface then almost the same analysis shows the vertices must provide extreme values. Graphically, three or more variables is difficult, however if you take a course in **Operations Research** it is likely you will learn the **simplex method** which provides an algebraic method to find the vertices through the introduction of so-called **slack variables**.

Example 5.4.5. *Let $f(x,y) = 2x^3 + y^4$. Find the absolute extrema of f on the unit-disk $D = \{(x,y) \mid x^2+y^2 \leq 1$.*

Solution: *note $\nabla f(x,y) = \langle 6x^2, 4y^3\rangle$ hence $(0,0)$ is the only critical point of f. Note $f(0,0) = 0$. Continuing, we analyze the boundary ∂D where $y^2 = 1-x^2$ hence*

$$f\big|_{\partial D}(x,y) = 2x^3 + (1-x^2)^2 = \underbrace{x^4+2x^3-2x^2+1}_{\text{let this be } g(x)}.$$

Note,

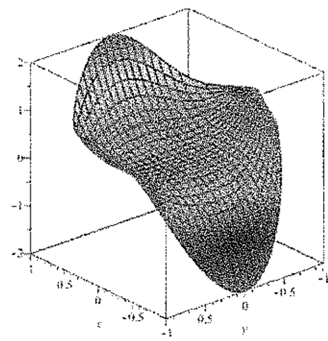
$$g'(x) = 4x^3+6x^2-4x = 2x(2x^2+3x-2) = 2x(2x-1)(x+2).$$

Thus $x = 0, 1/2, -2$ are critical numbers for g. Note $x^2+y^2 = 1$ yields points $(0,\pm 1)$, $(1/2, \pm\sqrt{3}/2)$ whereas $x = -2$ gives no solutions in the unit-circle. We calculate,

$$f(0,\pm 1) = 2(0)^3 + (\pm 1)^4 = 1 \quad \& \quad f(1/2, \pm\sqrt{3}/2) = 2(1/2)^3 + (\sqrt{3}/2)^4 = 13/16.$$

A subtle point[7] which matters to this problem, y is not a differentiable function of x on an open set centered about $x = \pm 1$. Note the points $(\pm 1, 0)$ are on the unit-circle and we obtain $f(1,0) = 2$ and $f(-1,0) = -2$. We find the maximum of f is 2 is attained at $(1,0)$. Whereas the mininum of f is -2 which is attained at $(-1,0)$. Below I plot $z = f(x,y)$ for $(x,y) \in D$:

[7] this is a good example of why you ignore the implicit function theorem to your own peril. Look at my old notes to see I speak from experience here.



To deal with ∂D in the problem above we could have studied $\nabla f = \lambda \nabla g$ for $g(x,y) = x^2 + y^2$ or we could have set $x = \cos t$ and $y = \sin t$ and sought out extreme values of $h(t) = 2\cos^3 t + \sin^4 t$. There are several ways to analyze the boundary in a given problem.

5.5 Problems

Problem 124 Suppose that the temperature T in the xy-plane changes according to

$$\frac{\partial T}{\partial x} = 8x - 4y \qquad \& \qquad \frac{\partial T}{\partial y} = 8y - 4x.$$

Find the maximum and minimum temperatures of T on the unit circle $x^2+y^2 = 1$. This time use the method of Lagrange multipliers. Hopefully we find agreement with Problem 107.

Problem 125 Use the method of Lagrange multipliers to find the point on the plane $x+2y-3z = 10$ which is closest to the point $(8,8,8)$.

Problem 126 Apply the method of Lagrange multipliers to solve the following problem: Let a, b be constants. Maximize xy on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Problem 127 Apply the method of Lagrange multipliers to solve the following problem: Find the distance from $(1,0)$ to the parabola $x^2 = 4y$.

Problem 128 Apply the method of Lagrange multipliers to solve the following problem: Suppose the base of a rectangular box costs twice as much per square foot as the sides and the top of the box. If the volume of the box must be $12ft^3$ then what dimensions should we build the box to minimize the cost?*[Please state the dimensions of the base and altitude clearly. Include a picture in your solution to explain the meaning of any variables you introduce, thanks!]*

Problem 129 Taking a break from the method of Lagrange. Assume a, b, c are constants: Show that the surfaces $xy = az^2$, $x^2 + y^2 + z^2 = b$ and $z^2 + 2x^2 = c(z^2 + 2y^2)$ are mutually perpendicular.

Problem 130 Apply the method of Lagrange multipliers to derive a formula for the distance from the plane $ax + by + cz + d = 0$ to the origin. If necessary, break into cases.

Problem 131 Suppose you want to design a soda can to contain volume V of soda. If the can must be a right circular cylinder then what radius and height should you use to minimize the cost of producing the can? *assume the cost is directly proportional to the surface area of the can*

Problem 132 Find any extreme values of xy^2z on the sphere $x^2 + y^2 + z^2$. *note the sphere is compact and the function $f(x, y, z) = xy^2z$ is continuous so this problem will have at least two interesting answers*

Problem 133 Again, breaking from optimization, this problem explores a concept some of you have not yet embraced. Find the point(s) on $x^2 + y^2 + z^2 = 4$ which the curve $\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$ intersects.

Problem 134 Consider $f(x, y) = x^3 - 3x - y^2$. Find any critical points for f and use the second derivative test for functions of two variables to judge if any of the critical points yield local extrema.

Problem 135 Consider $f(x, y) = x^2 - y^2$. Find any critical points for f and use the second derivative test for functions of two variables to judge if any of the critical points yield local extrema.

Problem 136 Consider $f(x, y) = x^3 + y^3 - 3xy$. Find any critical points for f and use the second derivative test for functions of two variables to judge if any of the critical points yield local extrema.

Problem 136+i An armored government agent decides to investigate a disproportionate use of electricity in a gated estate. Foolishly entering without a warrant he find himself at the mercy of Ron Swanson (at $(1, 0, 0)$), Dwight Schrute (at $(-1, 1, 0)$) and Kakashi (in a tree at $(1, 1, 3)$). Supposing Ron Swanson inflicts damage at a rate of 5 units inversely proportional from the square of his distance to the agent, and Dwight inflicts constant damage at a rate of 3 in a sphere of radius 2. If Kakashi inflicts a damage at a rate of 5 units directly proportional to the square of his distance from his location (because if you flee it only gets worse the further you run as he attacks you retreating) then where should you assume a defensive position as you call for back-up? What location minimizes your damage rate? Assume the ground is level and you have no jet-pack and/or antigravity devices.

Problem 137 Find global extrema for $f(x, y) = exp(x^2 - 2x + y^2 - 6y)$ on the closed region bounded by $x^2/4 + y^2/16 = 1$.

Problem 138 Find the maximum and minimum values for $f(x, y) = x^2 + y^2 - 1$ on the region bounded by the triangle with vertices $(-3, 0), (1, 4)$ and $(0, -3)$.

Problem 139 Find the maximum and minimum values for $f(x, y) = x^4 - 2x^2 + y^2 - 2$ on the closed disk with boundary $x^2 + y^2 = 9$.

Problem 140 Find the multivariate power series expansion for $f(x, y) = ye^x \sin(y)$ centered at $(0, 0)$

Problem 141 Expand $f(x, y, z) = xyz + x^2$ about the center $(1, 0, 3)$.

Problem 142 Given that $f(x, y) = 3 + 2x^2 + 3y^2 - 2xy + \cdots$ determine if $(0, 0)$ is a critical point and is $f(0, 0)$ a local extremum.

Problem 143 Use Clairaut's Theorem to show it is impossible for $\vec{F} = \langle y^3 + x, x^2 + y \rangle = \nabla f$.

Problem 144 Suppose $\vec{F} = \langle P, Q \rangle$ and suppose $P_y = Q_x$ for all points in some subset $U \subseteq \mathbb{R}^2$. Does it follow that $\vec{F} = \nabla f$ on U for some scalar function f? Discuss.

Hint: the polar angle θ has total differential $d\theta = d(\tan^{-1}(y/x)) = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dx$, think about the example $\vec{F} = \langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \rangle$. This function has domain $U = \mathbb{R}^2 - \{(0,0)\}$, can you find f such that $\vec{F} = \nabla f$ on all of U?

Problem 145 We say $U \subseteq \mathbb{R}^n$ is path-connected iff any pair of points in U can be connected by a polygonal-path (this is a path made from stringing together finitely many line-segments one after the other) . Show that if $\nabla f = 0$ on a path-connected set $U \subseteq \mathbb{R}^n$ then $f(\vec{x}) = c$ for each $\vec{x} \in U$. You may use the theorem from calculus I which states that if $f'(t) = 0$ for all t in a connected domain then $f = c$ on that domain.

Problem 146 Show that if $\nabla f = \nabla g$ on a path-connected set $U \subseteq \mathbb{R}^n$ then $f(\vec{x}) = g(\vec{x}) + c$ for each $\vec{x} \in U$. *Hint: you can use Problem 145.*

Problem 147 Prove the mean-value theorem for functions $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$. In particular, show that if f is differentiable at each point of the line-segment connecting $\vec{P}$ and $\vec{Q}$ then there exists a point $\vec{C}$ on the line-segment $\overline{PQ}$ such that $\nabla f(\vec{C}) \cdot (\vec{Q} - \vec{P}) = f(\vec{Q}) - f(\vec{P})$.
Hint: parametrize the line-segment and construct a function on $\mathbb{R}$ to which you can apply the ordinary mean value theorem, use the multivariate chain-rule and win.

Problem 148 The method of characteristics is one of the many calculational techniques suggested by the total differential. The idea is simply this: given $dx/dt = f(x,y)$ and $dy/dt = g(x,y)$ we can solve both of these for dt to eliminate time. This leaves a differential equation in just the cartesian coordinates x, y and we can usually use a separation of variables argument to solve for the level curves which the solutions to $dx/dt = f(x,y)$ and $dy/dt = g(x,y)$ parametrize. Use the technique just described to solve

$$\frac{dx}{dt} = -y \qquad \& \qquad \frac{dy}{dt} = x.$$

Problem 149 Suppose that the force $\vec{F} = q(\vec{v} \times \vec{B} + \vec{E})$ is the net-force on a mass m. Furthermore, suppose $\vec{B} = B\,\hat{z}$ and $\vec{E} = E\,\hat{z}$ where E and B are constants. Find the equations of motion in terms of the initial position $\vec{r}_o = \langle x_o, y_o, z_o \rangle$ and velocity $\vec{v}_o = \langle v_{ox}, v_{oy}, v_{oz} \rangle$ by solving the differential equations given by $\vec{F} = m\frac{d\vec{v}}{dt}$. If $E = 0$ and $v_{oz} = 0$ then find the radius of the circle in which the charge q orbits.
Hint: first solve for the velocity components via the technique from Problem 148 then integrate to get the components of the position vector.

Problem 150 Suppose objective function $f(x,y)$ has an extremum on $g(x,y) = 0$. Show that F defined by $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ recovers the extremum as a critical point. From this viewpoint, the adjoining of the multiplier converts the constrained problem in n-dimensions to an unconstrained problem in $(n+1)$-dimensions (you can easily generalize your argument to $n > 2$).