LECTURE 24: POYNTING'S THEOREM

Since $W_e = \frac{\varepsilon_0}{2}\int E^2 \, d\tau$ and $W_m = \frac{1}{2\mu_0}\int B^2 \, d\tau$

the work necessary for a total energy in the electric and magnetic fields is given by

$$W_e + W_m = \int \left(\frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) d\tau$$ hence,

$$\boxed{\text{Def}^{\underline{n}}/ \quad u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \quad \text{energy density of Electromagnetic field}}$$

Remark: you'll forgive me if I start thinking of $\vec{E}$ and $\vec{B}$ as two sides of the same coin, the electromagnetic field (Chapter 12 makes this manifest)

Suppose we have some charge and currents which produce $\vec{E}$ and $\vec{B}$ then in an instant dt the charge moves $d\vec{l}$ then the work done on q is

$$\begin{aligned} dW &= \vec{F} \cdot d\vec{l} \\ &= q(\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l} \\ &= q(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v} \, dt \\ &= q(\vec{E} \cdot \vec{v}) \, dt \end{aligned}$$

Hence, swapping $q \longmapsto \rho \, d\tau$ and $\rho \vec{v} \longmapsto \vec{J}$ we find for a collection of charge over a volume that,

$$\frac{dW}{dt} = \int_V (\vec{E} \cdot \vec{J}) \, d\tau$$

From Ampere's Law with Maxwell's Correction

$$\nabla \times \vec{B} = \mu_0\left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

We find $\vec{J} = \frac{1}{\mu_0}\nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ thus $\vec{E}\cdot\vec{J}$ is given by,

$$\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\underline{\vec{E}\cdot(\nabla \times \vec{B})} - \varepsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t}$$

Note,

$$\nabla\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{B})$$
$$= \vec{B}\cdot\left(-\frac{\partial \vec{B}}{\partial t}\right) - \vec{E}\cdot(\nabla\times\vec{B})$$
$$\therefore\ \underline{\vec{E}\cdot(\nabla\times\vec{B})} = -\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} - \nabla\cdot(\vec{E}\times\vec{B})$$

Therefore,

$$\vec{E}\cdot\vec{J} = -\varepsilon_0\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} - \frac{1}{\mu_0}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})$$
$$= -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})$$

Consequently,

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\vec{E}\cdot\vec{J})\,d\tau$$
$$= -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\int_{\mathcal{V}} \nabla\cdot(\vec{E}\times\vec{B})\,d\tau$$
$$= -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_{\partial\mathcal{V}} (\vec{E}\times\vec{B})\cdot d\vec{a}$$

$$\boxed{\text{Def}^{\underline{n}}/\ \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}\quad \text{Poynting vector}}$$

$$\boxed{\frac{dW}{dt} = \frac{-d}{dt}\int_{\mathcal{V}} u\,d\tau - \oint_{\partial\mathcal{V}} \vec{S}\cdot d\vec{a}}$$

Poynting vector $\underbrace{\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})}_{\text{energy flux density}}$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\partial\mathcal{V}} (\vec{E} \times \vec{B}) \cdot d\vec{a}$$

$$= \underbrace{-\frac{d}{dt}\int_{\mathcal{V}} u \, d\tau}_{\substack{\text{change in} \\ \text{total energy stored} \\ \text{in } \vec{E} \text{ \& } \vec{B} \text{ fields}}} - \oint_{\partial\mathcal{V}} \vec{S} \cdot d\vec{a}$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u \, d\tau - \int_{\mathcal{V}} (\nabla \cdot \vec{S}) \, d\tau$$

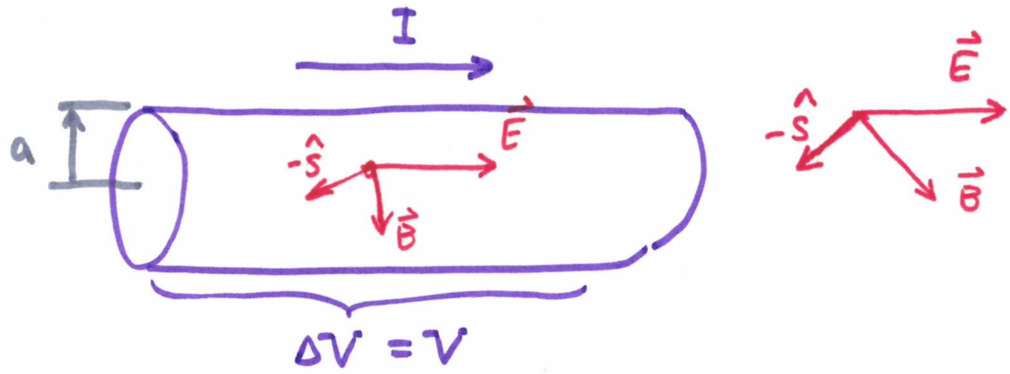
$$= -\int_{\mathcal{V}} \left(\frac{\partial u}{\partial t} + \nabla \cdot \vec{S}\right) d\tau$$

In empty space $\frac{dW}{dt} = 0$ (no charges in empty space)

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = 0}$$

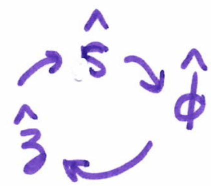
conservation of energy for electromagnetic field

E1 Suppose current flows down a resistive wire, then work is done and the wire heats up; $P = IV$ in the elementary course. Let's derive this via analysis of $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$ and Poynting's Th$^{\underline{m}}$



$$E = \frac{V}{L}$$ where L is length of cylindrical resistor

$$B = \frac{\mu_0 I}{2\pi a}$$



$$\vec{E} \times \vec{B} = \frac{\mu_0 I V}{2\pi a L} \hat{z} \times \hat{\phi}$$

$$\vec{E} \times \vec{B} = \frac{-\mu_0 I V}{2\pi a L} \hat{s}$$

$$\int \vec{S} \cdot d\vec{a} = \frac{-\mu_0 I V}{2\pi a L \mu_0} \int \hat{s} \cdot d\vec{a} = \frac{-\mu_0 I V}{\mu_0 (2\pi a L)} (2\pi a L)$$

$$\int \vec{S} \cdot d\vec{a} = -IV$$

$$\frac{dW}{dt} = -\int \vec{S} \cdot d\vec{a} = \boxed{IV}$$