LECTURE 24: MATRICES AND LINEAR MAPS

Def^n/ $A \in \mathbb{R}^{m \times n}$ is an $m \times n$ matrix, that is A has m-rows and n-columns. We say A_{ij} is the (i,j)-th entry of A.

$$A = \left[col_1(A) \,\middle|\, col_2(A) \,\middle|\, \cdots \,\middle|\, col_n(A) \right] = \begin{bmatrix} row_1(A) \\ \hline row_2(A) \\ \hline \vdots \\ row_m(A) \end{bmatrix}$$

Two matrices of the same size are equal iff all components match; $A = B \iff A_{ij} = B_{ij} \; \forall i,j$.
Furthermore, given $A, B \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}$

$$(A+B)_{ij} = A_{ij} + B_{ij} \qquad \text{defines } A+B \in \mathbb{R}^{m \times n}$$
$$(cA)_{ij} = cA_{ij} \qquad \text{defines } cA \in \mathbb{R}^{m \times n}.$$

If $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then $AB \in \mathbb{R}^{m \times p}$ is def^n by $(AB)_{ij} = \sum_k A_{ik} B_{kj} = row_i(A) \cdot col_j(B)$.

Def^n/ $L : \mathbb{R}^n \to \mathbb{R}^m$ is a <u>linear transformation</u> iff $L(cx+y) = cL(x) + L(y) \quad \forall x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Th^m/ $L : \mathbb{R}^n \to \mathbb{R}^m$, then L is linear transformation iff $\exists A \in \mathbb{R}^{m \times n}$ such that $L(x) = Ax \quad \forall x \in \mathbb{R}^n$

<u>Proof</u>: If $\exists A \in \mathbb{R}^{m \times n}$ s.t. $L(x) = Ax$ then if $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ note $L(cx+y) = A(cx+y) = cAx + Ay = cL(x) + L(y)$. thus L is linear map. Likewise if L is linear map then

$$L(x) = L\Big(\sum_i x_i e_i\Big) = \sum_i x_i L(e_i) = [L]x$$

for more details

thus identify $A = [L]$. //

Def^n/ $\mathbb{R}^n$ has <u>standard basis</u> $\{e_1, e_2, \dots, e_n\}$ where $(e_i)_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ and we may express $x = \sum x_i e_i$ for any $x \in \mathbb{R}^n$. Here, $x_i = x \cdot e_i$

<u>Remark</u>: If $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$ then

$$Ax = \underbrace{x_1 \, col_1(A) + x_2 \, col_2(A) + \cdots + x_n \, col_n(A)}_{\substack{\text{linear combination of columns} \\ \text{of } A \text{ with weights given} \\ \text{by components of } x}}$$

Def^n/ If $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear transformation then $[L] = [L(e_1) | L(e_2) | \cdots | L(e_n)]$ defines the <u>standard</u> <u>matrix</u> of L $\left([L] \in \mathbb{R}^{m \times n} \right)$

Returning to $\Rightarrow$) direction of Th^m once more, if L is linear transformation and $x = \sum_i x_i e_i \in \mathbb{R}^n$ then we calculate

$$\begin{aligned} L(x) &= \sum_{i=1}^{n} x_i \, L(e_i) \\ &= \sum_{i=1}^{n} x_i \, col_i([L]) \\ &= [L]x. // \end{aligned}$$

Example 1: $L(x,y) = (x+2y,\ 3x-4y) = \begin{bmatrix} x+2y \\ 3x-4y \end{bmatrix}$

$$\left.\begin{aligned} L(e_1) &= L(1,0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ L(e_2) &= L(0,1) = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \end{aligned}\right\} \quad [L] = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

Indeed $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x-4y \end{bmatrix}.$

Example 2: $L(x,y,z) = (2x,\ 3y,\ 4z)$

$$= \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{[L]}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Example 3: $L(x,y) = \begin{bmatrix} x+2y \\ 3x+4y \\ 5x+6y \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + y\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{[L]}\begin{bmatrix} x \\ y \end{bmatrix}$

$L: \mathbb{R}^2 \to \mathbb{R}^3$

Example 4: $L(x,y,z) = (x,\ 2y+z,\ 3z) = \begin{bmatrix} x \\ 2y+z \\ 3z \end{bmatrix}$

$$[L] = \big[L(e_1) \,|\, L(e_2) \,|\, L(e_3)\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 5: $L(x,y,z) = 3x+6y-z = \underbrace{[3,\ 6,\ -1]}_{[L]}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$