LECTURE 25: MAXWELL'S STRESS ENERGY TENSOR, FORCE & MOMENTUM

Force on charges in a volume $\mathcal{V}$ given by:

$$\vec{F} = \int_{\mathcal{V}} (\vec{E} + \vec{v}\times\vec{B})\rho\, d\tau = \int_{\mathcal{V}} (\rho\vec{E} + \vec{J}\times\vec{B})\, d\tau$$

Therefore,

$$\frac{d\vec{F}}{d\tau} = \vec{f} = \rho\vec{E} + \vec{J}\times\vec{B} \quad \leftarrow \text{Electromagnetic force per unit volume}$$

$$\vec{f} = \epsilon_0(\nabla\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$$

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E}$$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\nabla\times\vec{E})$$

Consequently,

$$\vec{f} = \epsilon_0\left[(\nabla\cdot\vec{E})\vec{E} - \vec{E}\times(\nabla\times\vec{E})\right] - \frac{1}{\mu_0}\left[\vec{B}\times(\nabla\times\vec{B})\right] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$= \underbrace{\epsilon_0\left[(\nabla\cdot\vec{E})\vec{E} - \underline{\vec{E}\times(\nabla\times\vec{E})}\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\vec{B})\vec{B} - \underline{\vec{B}\times(\nabla\times\vec{B})}\right]}_{\text{these terms can be reformatted with help of product rule (4)}} - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$$

$$\nabla E^2 = \nabla(\vec{E}\cdot\vec{E}) = \vec{E}\times(\nabla\times\vec{E}) + \vec{E}\times(\nabla\times\vec{E}) + (\vec{E}\cdot\nabla)\vec{E} + (\vec{E}\cdot\nabla)\vec{E}$$

$$\nabla E^2 = 2\vec{E}\times(\nabla\times\vec{E}) + 2(\vec{E}\cdot\nabla)\vec{E}$$

$$\therefore \quad \underline{\vec{E}\times(\nabla\times\vec{E})} = \frac{1}{2}\nabla E^2 - (\vec{E}\cdot\nabla)\vec{E}$$

and, $\underline{\vec{B}\times(\nabla\times\vec{B})} = \frac{1}{2}\nabla B^2 - (\vec{B}\cdot\nabla)\vec{B}$

$$\vec{f} = \epsilon_0\left[(\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E}\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B}\right]$$
$$- \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

This can be simplified by introducing MAXWELL'S STRESS TENSOR,

$$\boxed{\text{Def}^{\underline{n}}/\quad T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)}$$

I'm not sure it matters, but we can form the tensor via

$$T = \sum_{i,j=1}^{3} T_{ij}\, dx^i \otimes dx^j$$

and then,

$$T(\vec{v},\vec{w}) = \sum_{i,j=1}^{3} T_{ij}\, dx^i(\vec{v})\, dx^j(\vec{w})$$
$$= \sum_{i,j=1}^{3} T_{ij}\, v^i w^j$$
$$= [v^1, v^2, v^3]\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}\begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}$$

In particular, $T(\hat{x}_i, \hat{x}_j) = T_{ij} = \hat{x}_i^T [T]\hat{x}_j$

Remark: GRIFFITHS USES WEIRD NOTATION HERE, $\vec{a}\cdot\overleftrightarrow{T}$ and $\overleftrightarrow{T}\cdot\vec{a}$ and well, whatever.

Continuing, let's contract T_{ij} against $\partial_i = \frac{\partial}{\partial x_i}$

$$\sum_{i=1}^{3} \partial_i T_{ij} = \sum_{i=1}^{3} \partial_i \left[\epsilon_0 \left(E_i E_j - \tfrac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0}\left(B_i B_j - \tfrac{1}{2}\delta_{ij} B^2 \right) \right]$$

$$= \epsilon_0 \sum_{i=1}^{3} \left[(\partial_i E_i) E_j + E_i \partial_i E_j - \tfrac{1}{2}\delta_{ij} (2\vec{E}\,\nabla E) \right] +$$

$$+ \frac{1}{\mu_0} \sum_{i=1}^{3} \left[(\partial_i B_i) B_j + B_i \partial_i B_j - \tfrac{1}{2}\delta_{ij}\, \partial_i (\vec{B}\cdot\vec{B}) \right]$$

$$= \epsilon_0 \left[(\nabla\cdot\vec{E}) E_j + (\vec{E}\cdot\nabla) E_j \right] + \frac{1}{\mu_0}\left[(\nabla\cdot\vec{B}) B_j + (\vec{B}\cdot\nabla) B_j \right] +$$

$$- \epsilon_0 \sum_{i=1}^{3} \tfrac{1}{2}\delta_{ij}\, \partial_i (E^2) - \frac{1}{\mu_0} \sum_{i=1}^{3} \tfrac{1}{2}\delta_{ij}\, \partial_i B^2$$

$$= \epsilon_0 \left[(\nabla\cdot\vec{E}) E_j + (\vec{E}\cdot\nabla) E_j - \tfrac{1}{2}\partial_j (E^2) \right] +$$

$$+ \frac{1}{\mu_0}\left[(\nabla\cdot\vec{B}) B_j + (\vec{B}\cdot\nabla) B_j - \tfrac{1}{2}\partial_j (B^2) \right]$$

Oh, now sum over j and make it a vector,

$$\sum_{i,j=1}^{3} (\partial_i T_{ij})\hat{x}_j = \epsilon_0 \left[(\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E} - \tfrac{1}{2}\nabla(E^2) \right]$$
$$+ \frac{1}{\mu_0}\left[(\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B} - \tfrac{1}{2}\nabla(B^2) \right]$$

Therefore,

$$\boxed{\vec{f} = \underbrace{\sum_{i=1}^{3}\sum_{j=1}^{3} (\partial_i T_{ij})\hat{x}_j}_{\text{GRIFFITHS DENOTES THIS } \nabla\cdot\overleftrightarrow{T}} - \epsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B})}$$

Recall we defined Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ hence,

$$\boxed{\vec{f} = \sum_{i,j=1}^{3} (\partial_i T_{ij})\,\hat{x}_j - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}}$$

Now return to calculate force on charges in $\mathcal{V}$ due to the electromagnetic forces due to $\vec{E}$ and $\vec{B}$

$$\vec{F} = \int_{\mathcal{V}} \vec{f}\, d\tau$$

$$= \int_{\mathcal{V}} \left(\sum_{i,j=1}^{3} (\partial_i T_{ij})\,\hat{x}_j - \epsilon_0\mu_0 \frac{\partial \vec{S}}{\partial t} \right) d\tau$$

$$= \int_{\mathcal{V}} \nabla \cdot \sum_{j=1}^{3} T_{ij}\,\hat{x}_j$$

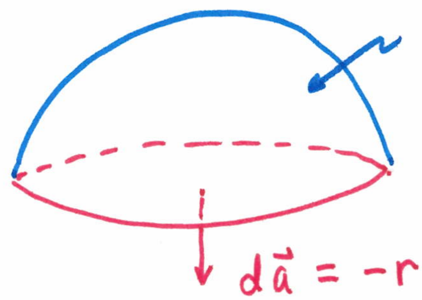
$$= \int_{\mathcal{V}} \left(\sum_{i=1}^{3} \partial_i \sum_{j=1}^{3} T_{ij}\,\hat{x}_j \right) d\tau - \epsilon_0\mu_0 \frac{d}{dt} \int_{\mathcal{V}} \vec{S}\, d\tau$$

$$= \underbrace{\int_{\partial\mathcal{V}} \left(\sum_{i,j=1}^{3} T_{ij}\,\hat{x}_j \right) \hat{x}_i \cdot d\vec{a}}_{\int_{\partial\mathcal{V}} \overleftrightarrow{T} \cdot d\vec{a}} - \epsilon_0\mu_0 \frac{d}{dt} \int_{\mathcal{V}} \vec{S}\, d\tau$$

$\int_{\partial\mathcal{V}} \overleftrightarrow{T} \cdot d\vec{a}$ is $\approx$ GRIFFITH'S NOTATION

<u>Remark</u>: $\boxed{\vec{F} = \int_{\partial\mathcal{V}} T(d\vec{a}, \;) - \mu_0\epsilon_0 \frac{d}{dt} \int_{\mathcal{V}} \vec{S}\, d\tau}$

$\boxed{E1}$ Determine net-force on northern hemisphere of a uniformly charged solid sphere of radius R and charge Q



$d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$\hat{r} = \langle \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \rangle$

$d\vec{a} = -r\,dr\,d\phi \, \hat{z}$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \hat{r} = \underbrace{\frac{Q}{4\pi\epsilon_0 R^2} \langle \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \rangle}_{\text{on the hemisphere top.}}$$

$$\vec{F} = \int_{\partial V} \left(\sum_{i,j=1}^{3} T_{ij} \hat{x}_j \right) \hat{x}_i \cdot d\vec{a} \Rightarrow F_z = \int_{\partial V} \sum_{i=1}^{3} T_{i3} \hat{x}_i \cdot d\vec{a}$$

$$F_z = \int_{\partial V} T_{13} \hat{x}_1 \cdot d\vec{a} + T_{23} \hat{x}_2 \cdot d\vec{a} + T_{33} \hat{x}_3 \cdot d\vec{a}$$

$$T_{13} = \epsilon_0 E_1 E_3 \qquad d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \langle \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \rangle$$

$$T_{23} = \epsilon_0 E_2 E_3$$

$$T_{33} = \epsilon_0 \left(E_3^2 - \frac{1}{2}(E_1^2 + E_2^2 + E_3^2) \right) = \epsilon_0 \left(E_3^2 - E_1^2 - E_2^2 \right) \frac{1}{2}$$

For the top hemisphere,

$$d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \langle \cos\phi \sin\theta, \ \sin\phi \sin\theta, \ \cos\theta \rangle$$

$$\hat{x}_1 \cdot d\vec{a} = R^2 \sin^2\theta \cos\phi \, d\theta \, d\phi$$

$$\hat{x}_2 \cdot d\vec{a} = R^2 \sin^2\theta \sin\phi \, d\theta \, d\phi$$

$$\hat{x}_3 \cdot d\vec{a} = R^2 \sin\theta \cos\theta \, d\theta \, d\phi$$

$$T_{13} = \epsilon_0 E_1 E_3 = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2} \cos\phi \sin\theta \right) \left(\frac{Q}{4\pi\epsilon_0 R^2} \cos\theta \right)$$

$$T_{13} = \frac{\epsilon_0 Q^2}{(4\pi\epsilon_0 R^2)^2} \cos\phi \sin\theta \cos\theta$$

$$\int_{\partial V} T_{13} \hat{x}_1 \cdot d\vec{a} = \int_0^{\pi/2} \int_0^{2\pi} \frac{\epsilon_0 Q^2}{(4\pi\epsilon_0 R^2)^2} (\cos\phi \sin\theta \cos\theta) (R^2 \sin^2\theta \cos\phi) \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \frac{\epsilon_0 Q^2 R^2}{(4\pi\epsilon_0 R^2)^2} \cos^2\phi \sin^3\theta \cos\theta \, d\phi \, d\theta$$

$$T_{23} = \epsilon_0 E_2 E_3 = \epsilon_0 \left(\frac{Q \sin\phi \sin\theta}{4\pi\epsilon_0 R^2} \right) \left(\frac{Q \cos\theta}{4\pi\epsilon_0 R^2} \right)$$

$$\int_{\partial V} T_{23} \hat{x}_2 \cdot d\vec{a} = \int_0^{\pi/2} \int_0^{2\pi} \frac{\epsilon_0 Q^2 R^2}{(4\pi\epsilon_0 R^2)^2} (\sin\phi \sin\theta \cos\theta) \sin^2\theta \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \frac{\epsilon_0 Q^2 R^2}{(4\pi\epsilon_0 R^2)^2} \sin^2\phi \sin^3\theta \cos\theta \, d\phi \, d\theta$$

$$T_{33} = \tfrac{1}{2}\epsilon_0 \left(E_3^2 - E_1^2 - E_2^2 \right)$$

$$= \frac{\epsilon_0}{2} \left(\frac{Q}{4\pi\epsilon_0 R^2} \right)^2 \left[\cos^2\theta - \cos^2\phi \sin^2\theta - \sin^2\phi \sin^2\theta \right]$$

$$= \frac{\epsilon_0}{2} \left(\frac{Q}{4\pi\epsilon_0 R^2} \right)^2 \left[\cos^2\theta - \sin^2\theta \right]$$

$$\int_{\partial V} T_{33} \hat{x}_3 \cdot d\vec{a} = \int_0^{\pi/2} \int_0^{2\pi} \frac{\epsilon_0}{2} \left(\frac{Q}{4\pi\epsilon_0 R^2} \right)^2 [\cos^2\theta - \sin^2\theta] \cos\theta \cdot R^2 \sin\theta \, d\phi \, d\theta$$

continuing the top hemisphere

$$F_z^{\textcircled{I}} = \int_{\partial \mathcal{V}_{top}} \left(T_{13}\, \hat{x}_1 \cdot d\vec{a} + T_{23}\, \hat{x}_2 \cdot d\vec{a} + T_{33}\, \hat{x}_3 \cdot d\vec{a} \right)$$

$$= \frac{\varepsilon_0 Q^2 R^2}{(4\pi \varepsilon_0 R^2)^2} \int_0^{\pi/2} \int_0^{2\pi} \Big[\underline{\cos^2\phi} \sin^3\theta \cos\theta + \underline{\sin^2\phi} \sin^3\theta \cos\theta + \frac{1}{2}(\cos^2\theta - \sin^2\theta)\cos\theta \sin\theta \Big] d\phi\, d\theta$$

$$= \frac{\varepsilon_0 Q^2 R^2}{2(4\pi \varepsilon_0 R^2)^2} \int_0^{\pi/2} \int_0^{2\pi} \left[2\sin^3\theta \cos\theta + (\cos^2\theta - \sin^2\theta)\cos\theta \sin\theta \right] d\phi\, d\theta$$

$$= \frac{\varepsilon_0 Q^2 R^2}{2(4\pi \varepsilon_0 R^2)^2} (2\pi) \int_0^{\pi/2} \left(\sin^3\theta \cos\theta + \cos^3\theta \sin\theta \right) d\theta$$

$$= \frac{\varepsilon_0 Q^2 R^2}{2(4\pi \varepsilon_0 R^2)^2} (2\pi) \int_0^{\pi/2} \left(\sin^2\theta + \cos^2\theta \right) \sin\theta \cos\theta \, d\theta$$

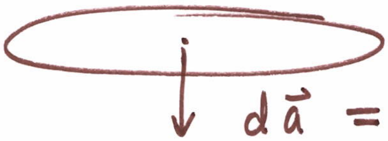
$$= \frac{\pi \varepsilon_0 Q^2 R^2}{(4\pi \varepsilon_0 R^2)^2} \int_0^{\pi/2} \frac{1}{2} \sin(2\theta)\, d\theta$$

$$= \frac{\pi \varepsilon_0 Q^2 R^2}{(4\pi \varepsilon_0 R^2)^2} \left(\frac{-1}{4} \cos(2\theta) \right|_0^{\pi/2}$$

$$= \frac{\pi \varepsilon_0 Q^2 R^2}{2(4\pi \varepsilon_0 R^2)^2}$$

$$= \frac{1}{4\pi \varepsilon_0} \frac{Q^2}{8R^2} \qquad (\text{Eq}^{\underline{n}}\ 8.23 \text{ in GRIFFITHS})$$

Continuing, calculate $\int_{\partial V_{DISK}} T(d\vec{a}, \ )$ where



$$d\vec{a} = -r\,dr\,d\phi\,\hat{z} \qquad (s = r \text{ on } z = 0)$$

$$\vec{E} = \left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)\vec{r} = \frac{Q}{4\pi\varepsilon_0 R^3}\langle r\cos\phi, r\sin\phi, 0\rangle$$

$$E_1 = \frac{Qr\cos\phi}{4\pi\varepsilon_0 R^3} \ \ \& \ \ E_2 = \frac{Qr\sin\phi}{4\pi\varepsilon_0 R^3} \ \ \& \ \ E_3 = 0$$

Then since $\hat{x}_1 \cdot d\vec{a} = 0$ and $\hat{x}_2 \cdot d\vec{a} = 0$ we face,

$$F_z^{\text{II}} = -\int_{\partial V_{disk}} T_{33}\, r\,dr\,d\phi$$

$$T_{33} = \frac{\varepsilon_0}{2}\left(E_3^2 - E_1^2 - E_2^2\right) = \frac{\varepsilon_0}{2}\left(\frac{Qr}{4\pi\varepsilon_0 R^3}\right)^2\left(-\cos^2\phi - \sin^2\phi\right)$$

$$= -\int_0^{2\pi}\int_0^R -\frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)^2 r^2\, r\,dr\,d\phi$$

$$= \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)^2 (2\pi)\left(\frac{R^4}{4}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q^2}{16R^2}\right)$$

Consequently,

$$F_z = F_z^{\text{I}} + F_z^{\text{II}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q^2}{8R^2} + \frac{Q^2}{16R^2}\right)$$

$$F_z = \frac{3Q^2}{4\pi\varepsilon_0(16R^2)} = \boxed{\frac{3Q^2}{64\pi\varepsilon_0 R^2}}$$