Lecture 27: The Wave Equation for Electromagnetism

Maxwell's Equations in empty space are simply,

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Recall that $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$ therefore,

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = -\nabla^2 \vec{B}$$

Consequently, by Faraday's Law and Maxwell's correction,

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\nabla^2 \vec{E}$$

$$\nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\nabla^2 \vec{B}$$

Hence,

$$-\frac{\partial}{\partial t}\left(\nabla \times \vec{B}\right) = -\nabla^2 \vec{E} \Rightarrow -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\nabla^2 \vec{E}$$

$$\mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\nabla \times \vec{E}\right) = -\nabla^2 \vec{B} \Rightarrow \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right) = -\nabla^2 \vec{B}$$

Therefore,

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$\vec{E}$ & $\vec{B}$ solve a wave eq$^{\underline{n}}$ with speed $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$$\boxed{\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{(\sqrt{\mu_0 \epsilon_0})^2} \nabla^2 \vec{E} \quad \text{and} \quad \frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{(\sqrt{\mu_0 \epsilon_0})^2} \nabla^2 \vec{B}}$$