Lecture 27: Linear Algebra, Eigenvectors

Def$^{\underline{n}}$/ If $A \in \mathbb{R}^{n\times n}$ and $\exists \lambda \in \mathbb{R}$ and $v \neq 0$ in $\mathbb{R}^n$ such that $Av = \lambda v$ then v is <u>eigenvector</u> with <u>eigenvalue</u> λ.

Observe $Av = \lambda v \Leftrightarrow (A - \lambda I)v = 0$
thus $v \neq 0$ with $Av = \lambda v \Leftrightarrow (A - \lambda I)v = 0$ with $v \neq 0$
But, $(A - \lambda I)(0) = 0$ thus $(A - \lambda I)^{-1}$ d.n.e.
and we find $\det(A - \lambda I) = 0$.

Def$^{\underline{n}}$/ Given $A \in \mathbb{R}^{n\times n}$ the <u>characteristic eq$^{\underline{n}}$</u> is $\det(A - \lambda I) = 0$. Sol$^{\underline{n}}$'s to the characteristic equation are characteristic sol$^{\underline{n}}$'s ... these are eigenvalues.

Th$^{\underline{m}}$/ Given $A \in \mathbb{R}^{n\times n}$ the eigenvalues of A are solutions of $\det(A - \lambda I) = 0$. For each sol$^{\underline{n}}$ there exists at least one eigenvector corresponding the sol$^{\underline{n}}$.

Generally the story is complicated for $A \in \mathbb{R}^{n\times n}$, there may be no eigenvalues, the eigenvectors may be insufficient to form an eigenbasis for $\mathbb{R}^n$. To understand the general problem we're led to the <u>REAL JORDAN FORM</u>.

For example, to solve $\frac{d\vec{r}}{dt} = A\vec{r}$ we may write $\vec{r} = e^{tA}\vec{r}_o$ where $e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \cdots$. It turns out $A = P^{-1}JP$ where J = nice Jordan Form of A and $\exp(tA) = \exp(P^{-1}tJP) = P^{-1}\exp(tJ)P$ ← way to calculate w/o series madness...

• Oh, so we don't need all this here... ?

Th^m (Real Spectral Theorem)

If $A^T = A$ then the eigenvalues of A are real and A is orthonormally diagonalizable.

Let me unpack what is meant above. Given $A^T = A$,

1. $\det(A - \lambda I) = 0$ has sol^{n}'s $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ (possible repeats)

2. $\exists$ orthonormal vectors $\{v_1, v_2, \dots, v_n\}$ means $v_i \cdot v_j = \delta_{ij}$ $\forall i,j = 1,2,\dots,n$ with $Av_i = \lambda_i v_i$ $\forall i = 1,2,\dots,n$

3. If $P = [v_1 | v_2 | \cdots | v_n]$ then $P^{-1} = P^T$ by 2 and we calculate,

$$\begin{aligned}
P^{-1}AP &= P^T A [v_1 | v_2 | \cdots | v_n] \\
&= P^T [Av_1 | Av_2 | \cdots | Av_n] \\
&= P^T [\lambda_1 v_1 | \lambda_2 v_2 | \cdots | \lambda_n v_n] \\
&= \begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline \vdots \\ v_n^T \end{bmatrix} [\lambda_1 v_1 | \lambda_2 v_2 | \cdots | \lambda_n v_n] \\
&= \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad (v_i^T v_j = v_i \cdot v_j = \delta_{ij})
\end{aligned}$$

Remark: our main interest is applying this math to the Hessian in the multivariate Taylor Expansion. In short, it is based on a symmetric matrix so... what I share ↴ is of utmost importance.

Def^n/ If $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ can be expressed as $Q(x) = x^T A x$ for a symmetric matrix A then Q is a <u>quadratic form</u>.

<u>Example 1</u>: $Q(x,y) = x^2 + 4xy + 2y^2 = [x, y]\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

<u>Example 2</u>: $Q(x,y,z) = x^2 - y^2 + 7z^2 + 3xy = [x, y, z]\begin{bmatrix} 1 & 3/2 & 0 \\ 3/2 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Def^n/ If $Q(x) = x^T A x$ then $[Q] = A$ is the <u>standard matrix</u> of the <u>quadratic form</u> Q

Th^m/ Given a quadratic form $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ there exist eigen coordinates $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ such that

$$Q(x) = \lambda_1 \bar{x}_1^2 + \lambda_2 \bar{x}_2^2 + \cdots + \lambda_n \bar{x}_n^2$$

In fact, $\bar{x} = P^T x$ where $P = [v_1 | v_2 | \cdots | v_n]$ for orthonormal $\{v_i\}$ eigenbasis of $[Q]$.

≈Proof

$$\begin{aligned} Q(x) = x^T A x &= x^T P P^T A P P^T x \qquad : PP^T = I = PP^T \\ &= (P^T x)^T (P^T A P)(P^T x) \\ &= \bar{x}^T \, \mathrm{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n)\, \bar{x} \\ &= \lambda_1 \bar{x}_1^2 + \lambda_2 \bar{x}_2^2 + \cdots + \lambda_n \bar{x}_n^2 . \end{aligned}$$

<u>Example 3</u>: $Q(x,y) = 2xy \Rightarrow [Q] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $\det([Q] - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$

Thus $\lambda_1 = -1$, $\lambda_2 = 1$ hence

$$Q(x,y) = 2xy = \underline{-\bar{x}_1^2 + \bar{x}_2^2}$$

Remark: we can now see $Q(x,y) = 1$ is a hyperbola

(8.) (40pts) Let $Q(x, y) = 2x^2 + 2xy + 2y^2$.

(a.) (8pts) Find a matrix A such that $Q(v) = v^T Av$.

(b.) (20pts) Find eigenvalues of A and unit-length eigenvectors u_1, u_2 for A

(c.) (8pts) What is the formula for $Q(v)$ where $v = \bar{x}u_1 + \bar{y}u_2$?

(d.) (4pts) Sketch the graph of $Q(x, y) = 9$

(a.) $Q(x,y) = [x, y]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$; $\boxed{A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$

(b.) $\det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (\lambda - 2)^2 - 1 = (\lambda - 3)(\lambda - 1)$

$\underline{\lambda_1 = 1}$ $A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \underline{u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$.

$\underline{\lambda_2 = 3}$ $A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \underline{u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

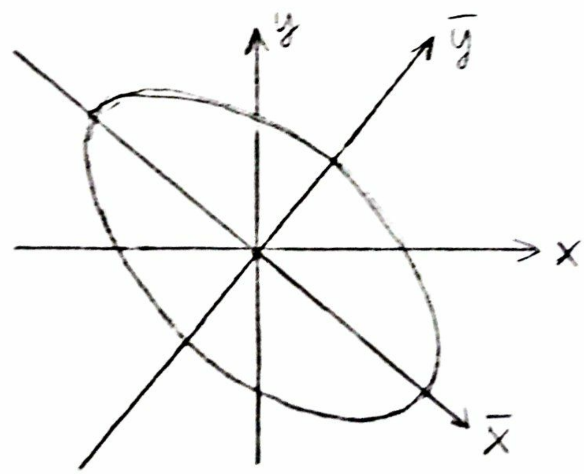
(c.) $Q(\bar{x}u_1 + \bar{y}u_2) = (\bar{x}u_1 + \bar{y}u_2)^T A(\bar{x}u_1 + \bar{y}u_2)$

$= \bar{x}^2 u_1^T A u_1 + \bar{x}\bar{y}(u_2^T A u_1 + u^T A u_2) + \bar{y}^2 u_2^T A u_2$

$= \bar{x}^2 u_1^T u_1 + \bar{x}\bar{y}(\underbrace{u_2^T u_1 + 3u_1^T u_2}_{0}) + \bar{y}^2 u_2^T(3u_2)$

$= \boxed{\bar{x}^2 + 3\bar{y}^2}$

(d.)



$\bar{x}^2 + 3\bar{y}^2 = 9$

$\underbrace{\frac{\bar{x}^2}{9} + \frac{\bar{y}^2}{3} = 1}_{\text{ellipse}}$

Remark: $2x^2 + 2xy + 2y^2 = 9$ is not obviously an ellipse.

P1#6 $Q(x,y) = 5x^2 + 5y^2 + 8xy$

$$Q(v) = v^T \underbrace{\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}_{A} v$$

1.) Calculate Eigenvalues of A.

$$\det(A - \lambda I) = \det\begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix} = (\lambda - 5)^2 - 4^2$$
$$= (\lambda - 9)(\lambda - 1)$$
$$\Rightarrow \underline{\lambda_1 = 1} \ \& \ \underline{\lambda_2 = 9}$$

2.) Find orthonormal e-basis for A,

$$A - I = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \Rightarrow u_1 = \frac{1}{\sqrt{2}}(1,-1)$$

$$A - 9I = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \Rightarrow u_2 = \frac{1}{\sqrt{2}}(1,1)$$

Remark: in principle 2×2 symmetric is extra nice because spectral Th^m implies $\mathcal{E}_1 \perp \mathcal{E}_2$ so you know $\mathcal{E}_1 = \operatorname{span}\{u_1\}$ & $\mathcal{E}_2 = \operatorname{span}\{u_2\}$ have $u_1 \perp u_2$. Once we find u_1 we can simply rotate $90°$ to find u_2!

$$\beta = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \therefore \quad [\beta] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \in SO(2).$$

3.) construct eigencoordinates $\bar{x}, \bar{y}$ s.t. $(x,y) = \bar{x}u_1 + \bar{y}u_2 = v$
$$= [u_1 | u_2]\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = [\beta]^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = [\beta]^T\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \boxed{\begin{aligned} \bar{x} &= \tfrac{1}{\sqrt{2}}(x - y) \\ \bar{y} &= \tfrac{1}{\sqrt{2}}(x + y) \end{aligned}}$$

these provide $\underline{Q(v) = \bar{x}^2 + 9\bar{y}^2}$.

$\boxed{P147}$ Writing $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we find,

$$Q(x,y,z) = 5x^2 + 5y^2 + 2z^2 + 8xy + 4xz + 4yz = v^t \underbrace{\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}}_{A} v$$

It was given A has $\lambda_1 = 1$, $\lambda_2 = 10$

Hence work towards finding orthonormal bases for $\mathcal{E}_1 = \text{Null}(A - I)$ and $\mathcal{E}_2 = \text{Null}(A - 10I)$

1.) $A - I = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $(u,v,w) \in \mathcal{E}_1$ has $w = -2u - 2v$

$\therefore \mathcal{E}_1 = \text{span}\{\underbrace{(1,0,-2)}_{u_1}, \underbrace{(0,1,-2)}_{u_2}\}$

Run G.S.A. on $\{u_1, u_2\}$,

$$u_1'' = \frac{1}{\sqrt{5}}(1,0,-2)$$

$$u_2' = u_2 - (u_2 \cdot u_1'')u_1'' = (0,1,-2) - (4/5)(1,0,-2)$$

$$u_2' = (-4/5, 1, -2 + 8/5) = \left(\frac{-4}{5}, \frac{5}{5}, \frac{-2}{5}\right) = \frac{1}{5}(-4,5,-2)$$

$$\therefore u_2'' = \frac{1}{\sqrt{45}}(-4,5,-2)$$

Then $\mathcal{E}_1 = \text{span}\{u_1'', u_2''\}$ as above.

2.) $A - 10I = \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} -20 & 16 & 8 \\ 20 & -25 & 10 \\ 20 & 20 & -80 \end{bmatrix} \sim \begin{bmatrix} -20 & 16 & 8 \\ 0 & -9 & 18 \\ 0 & 36 & -72 \end{bmatrix}$

$\sim \begin{bmatrix} -20 & 16 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -20 & 0 & 40 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $(u,v,w) \in \mathcal{E}_2$ has $u = 2w$, $v = 2w$

Hence $\mathcal{E}_2 = \text{span}\{\underbrace{\tfrac{1}{3}(2,2,1)}_{u_3''}\}$.

3.) $\underbrace{\beta = \left\{ \frac{1}{\sqrt{5}}(1,0,-2), \frac{1}{\sqrt{45}}(-4,5,-2), \frac{1}{3}(2,2,1) \right\}}_{\text{orthonormal eigenbasis for } A}$

P 147 continued

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = [\beta]^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [\beta]^{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as $[\beta]^{T} = [\beta]^{-1}$ for orthonormal basis.

Thus

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}}[1, 0, -2] \\ \frac{1}{3\sqrt{5}}[-4, 5, -2] \\ \frac{1}{3}[2, 2, 1] \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}}(x - 2z) \\ \frac{1}{3\sqrt{5}}(-4x + 5y - 2z) \\ \frac{1}{3}(2x + 2y + z) \end{bmatrix}$$

That is,

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{\sqrt{5}}(x - 2z) \\ \bar{y} &= \frac{1}{3\sqrt{5}}(-4x + 5y - 2z) \\ \bar{z} &= \frac{1}{3}(2x + 2y + z) \end{aligned}}$$

these provide,

$$Q(\bar{x}u_1'' + \bar{y}u_2'' + \bar{z}u_3'') = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2$$