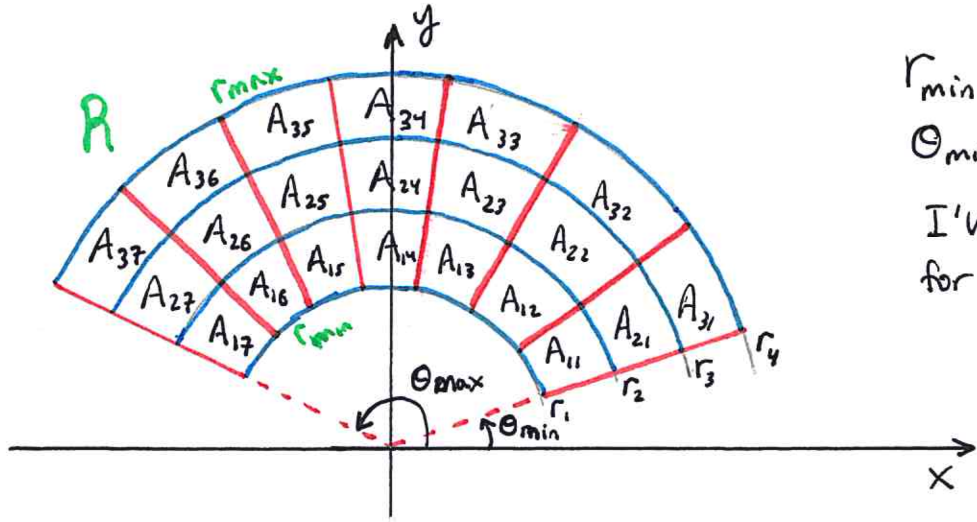
# LECTURE 27a: CHANGE OF VARIABLES pgs. 295-307 of 2010 notes.

I'll illustrate the Th$^{m}$ for $(n=2)$ by setting up an integral of $f(x,y)$ over a region natural to polar coordinates,




$$r_{min} \leq r \leq r_{max}$$

$$\theta_{min} \leq \theta \leq \theta_{max}$$

I've partitioned by $m=4$ for $r$ and $n=7$ for $\theta$

$$\Delta r = \frac{r_{max}-r_{min}}{3}$$

$$\Delta\theta = \frac{\theta_{max}-\theta_{min}}{7}$$

The area of the $(ij)$-th sector is $\Delta A_{ij} = \frac{1}{2}(r_{i+1}^2 - r_i^2)\Delta\theta$

↑ oops ☺ I had $\pi$ but since $\theta$ is in radians there is no $\pi$ here.

However, $r_{i+1} = r_i + \Delta r$ thus $r_{i+1}^2 - r_i^2$ gives,

$$\begin{aligned} r_{i+1}^2 - r_i^2 &= (r_i+\Delta r)^2 - r_i^2 \\ &= r_i^2 + 2r_i\Delta r + (\Delta r)^2 - r_i^2 \\ &= 2r_i\Delta r + (\Delta r)^2 \end{aligned}$$

Therefore, $\Delta A_{ij} = \frac{1}{2}\left(2r_i\Delta r + (\Delta r)^2\right)\Delta\theta \approx r_i\Delta r\Delta\theta$

The signed-volume over $z=f(x,y)$ is approximated by,

$$\iint_R f\,dA \cong \sum_{i=1}^{3}\sum_{j=1}^{7} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij} = \sum_{i=1}^{4}\sum_{j=1}^{7} f(x(r_{ij},\theta_{ij}), y(r_{ij},\theta_{ij}))\, r_i\Delta r\Delta\theta$$

$(r_{ij},\theta_{ij}) \in A_{ij}$

Then, as $\Delta r, \Delta\theta \to 0$ with $m,n\to\infty$ we obtain,

$$\iint_R f\,dA = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x(r_{ij},\theta_{ij}), y(r_{ij},\theta_{ij}))\, r_i\Delta r\Delta\theta$$

$$\Rightarrow \boxed{\iint_R f\,dA = \iint_{[r_{min},r_{max}]\times[\theta_{min},\theta_{max}]} f(x(r,\theta), y(r,\theta))\, r\,dr\,d\theta}$$

Heuristically:

$$dA = dx\,dy$$
$$dA = r\,dr\,d\theta$$

*these are equivalent views of the integration region. Therefore, by Fubini's Theorem:*

$$\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f\,dz\,dy\,dx &= \int_0^1 \int_0^{y^2} \int_0^{1-y} f\,dz\,dx\,dy \\
&= \int_0^1 \int_0^{1-z} \int_0^{y^2} f\,dx\,dy\,dz \\
&= \int_0^1 \int_0^{1-y} \int_0^{y^2} f\,dx\,dz\,dy \\
&= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f\,dy\,dz\,dx \\
&= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f\,dy\,dx\,dz.
\end{aligned}$$

**Remark 6.3.15.**

We have studied how to integrate in Cartesian Coordinates in some detail. It turns out that this is quite limiting. To do many interesting problems with better efficiency it pays to employ cylindrical or spherical coordinates. Before getting to those special choices we consider a general coordinate change briefly and in the process derive what we later use for the cylindrical and spherical coordinates.

## 6.4 Change of Variables in Multivariate Integration

Our goal in this section is to give partial motivation for the analog of $u$-substitution for multiple integrals. In the one-dimensional case, we have to substitute the new variables for the old in the integrand, change $dx$ to a corresponding expression with $du$, and we must change the bounds to the $u$-domain. We expect all three of these to appear in the multiple integral problem. It turns out the problem of substitution and bound-changing is nearly the same as in one-dimension. However, the method to change the measure offers a little surprise.

### 6.4.1 coordinate change and transformation

To begin, I encourage the reader to revisit Section 1.6 where we introduced polar, cylindrical and spherical coordinates. In general, a coordinate change on $\mathbb{R}^n$ is given by some mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ which is locally invertible at most points. Sometimes this is written as:

$$x_1 = x_1(u_1, u_2, \dots, u_n), \;\; x_2 = x_2(u_1, u_2, \dots, u_n), \;\; \dots \;\; x_n = x_n(u_1, u_2, \dots, u_n)$$

Invertibility means we can also solve for $u_i$ as a function of $x_i$:

$$u_1 = u_1(x_1, x_2, \dots, x_n), \;\; u_2 = u_2(x_1, x_2, \dots, x_n), \;\; \dots \;\; u_n = u_n(x_1, x_2, \dots, x_n).$$

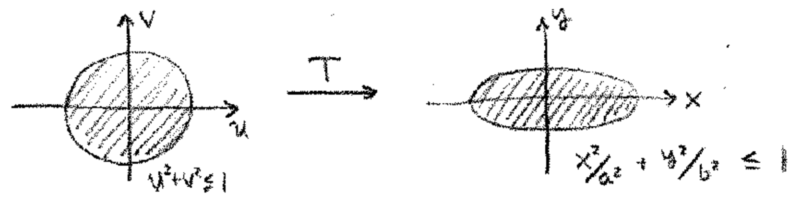
We don't insist on strict injectivity since polar, spherical and cylindrical coordinates all lack injectivity in their usual use. In particular, here are the usual polar coordinate transformations as well as the inverse transformations which apply to the half-space $x > 0$

$$x = r\cos\theta, \;\; y = r\sin\theta \qquad \& \qquad r = \sqrt{x^2+y^2}, \;\; \theta = \tan^{-1}(y/x).$$

We prefer to identify angles which correspond geometrically and the origin has $\rho = 0$ yet the polar and azmuthial angle of the origin is undefined. Fortunately, we are primarily interested in integration and the ambiguities of these standard coordinate systems do not change integrals. For a double integral, the addition or removal of a one-dimensional space does not change the integral. For a triple integral, the addition or removal of a two-dimensional space does not change the integral. To verify these assertions in general is beyond this course and properly belongs to the topic of **measure theory**. In addition, the more careful concept of a **coordinate chart** belongs to manifold theory where injectivity is required and some examples we consider here no longer fit the abstract definition of coordinate system.

We primarily discuss the problem of coordinate change in $\mathbb{R}^2$. It is a fortunate fact that the higher-dimensional problem admits almost the same analysis so little is lost by focusing on this readily visualized case. Let us begin with an example:

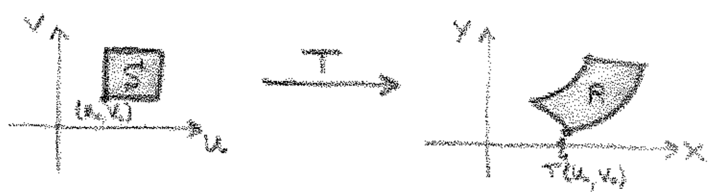
**Example 6.4.1.** *$S = \{(u,v) \mid u^2 + v^2 \leq 1\}$ and $x = au, y = bv$. Notice $x^2/2^2 + y^2/b^2 = a^2u^2/a^2 + b^2v^2/b^2 = u^2 + v^2 \leq 1$. Thus if we define $T(u,v) = (au, bv)$ we find*




*the transformations $T$ deforms a disk to an oval.*

### 6.4.2 determinants for good measure

Consider two planes, the $(x,y)$-plane and the $(u,v)$-plane, the coordinate change map $T$ takes $(u,v)$ to $T(u,v) = (x(u,v), y(u,v))$. In particular, we study $T : S \subset \mathbb{R}^2 \to R \subset \mathbb{R}^2$ and we insist that $T$ be invertible, except possibly on the boundaries. This means the equation relating $x, y$ and $u, v$ can be solved for either $x, y$ or $u, v$ locally. In the diagram below I illustrate how $T$ might map a rectangle in the $uv$-space to a curved region in $xy$-space.




However, if we focus on a very small region then a little rectangle is essentially sent to a little parallelogram. Relating the areas of the rectangle and parallelogram will provide the relation between the measure in $xy$ versus $uv$-coordinates. Furthermore, since we consider very small rectangle the first order approximation of $T$ suffices. Recall, in our discussion of differentiability we learned $T$ may be approximated by the linearization[7] of $T$

$$h(u,v) = T(u_o, v_0) + \begin{bmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{bmatrix} \begin{bmatrix} u - u_o \\ v - v_o \end{bmatrix}$$

[7]technically, this is an affine approximation of $T$

where the $2 \times 2$ Jacobian Matrix is evaluated at $(u_o, v_o)$. A parallelogram at $(u_o, v_o)$ is transported by $h$ to a new parallelogram at $T(u_o, v_o)$ whose area is distorted according to the structure of the Jacobian matrix. You can prove the following for some extra credit if you wish:

**Proposition 6.4.2.** *(based on 5.1 of Colley's Vector Calculus )*

Let $h(u,v) = \begin{bmatrix} x_o \\ y_o \end{bmatrix} + A \begin{bmatrix} u \\ v \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A \neq 0$ and $x_o, y_o$ are constants, then if $D^*$ is a parallelogram then $h(D^*) = D$ is also a parallelogram and

$$\text{Area}(D) = |det\ A|\text{Area}(D^*).$$

If $T$ maps $\triangle u, \triangle v$ at $(u_o, v_o)$ to $\triangle x, \triangle y$ at $T(u_o, v_o)$ then as $\triangle u, \triangle v \to 0$ the proposition above applied to the linearization yields:

$$\triangle x \, \triangle y = \det \begin{bmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{bmatrix} \triangle u \, \triangle v$$

For finite changes this is an approximation since the real changes in $x$ and $y$ respective are based on the generally non-linear nature of $T$. However, as the size of the rectangles is reduced the approximation improves. We ultimately apply the boxed formula inside an integral where the approximating rectangles are made arbitrarily small and consequently the result above is made exact. Of course, I have omitted some careful analysis here. There are many excellent advanced calculus texts which justify the multivariate change of variables theorem. For example, you might look at Munkres *Calculus on Manifolds*. It contains a lengthy justification of multivariate $u$-substitution. Finally, let us conclude with the generalization to $n$-dimensions. We again find the determinant of the Jacobian matrix gives the necessary volume rescaling factor. In particular, if $T(u_1, \dots u_n) = (x_1, \dots, x_n)$ is an invertible coordinate map which maps $\triangle u_1, \triangle u_2, \dots, \triangle u_n$ to $\triangle x_1, \triangle x_2, \dots, \triangle x_n$ then the volumes[8] of the parallel-$n$-pipeds are related by:

$$\triangle x_1 \triangle x_2 \cdots \triangle x_n = \det(DT) \triangle u_1 \triangle u_2 \cdots \triangle u_n$$

where $\det(DT)$ is the the determinant of the Jacobian matrix. For example, if $n = 3$ then

$$\triangle x \triangle y \triangle z = \det \begin{bmatrix} \partial_u x & \partial_v x & \partial_w x \\ \partial_u y & \partial_v y & \partial_w y \\ \partial_u z & \partial_v z & \partial_w z \end{bmatrix} \triangle u \triangle v \triangle w.$$

You may find the rows and columns of the matrix above reversed in some calculus texts. That operation of changing rows into corresponding columns is called **transposition**. You should learn in your linear algebra course that $\det A = \det A^T$ where $A^T$ is the **transpose** of $A$. It follows the formula above can also be written with the transpose of the Jacobian matrix. In any event, you should appreciate this section gives (without proof) an indication of the geometric significance of the determinant; the determinant quantifies generalized volume. The work which follows from here is merely a synthesis of the geometry of determinants with calculus.

[8] to be careful, the magnitude of these expressions are volumes, it is possible the expression is negative in which case the volume is given by the absolute value of the expression.

## 6.5 double integrals involving coordinate change

It is convenient to set some standard notation which helps us set-up the calculation of the Jacobian.

**Definition 6.5.1.**

The Jacobian of the transformation $T(u,v) = (x(u,v), y(u,v))$ is,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = x_u y_v - x_v y_u$$

The "Jacobian" is the determinant of what I called the "Jacobian matrix".

**Example 6.5.2.** *Consider polar coordinates: $T(r,\theta) = (r\cos\theta, r\sin\theta)$. Let's calculate the Jacobian, note $x = r\cos\theta$ and $y = r\sin\theta$*

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

**Example 6.5.3.** *Find Jacobian of the following transformation*

$$\begin{aligned} x &= 5u - v \quad \Rightarrow \quad x_u = 5,\ x_v = -1 \\ y &= u + 3v \quad \Rightarrow \quad y_u = 1,\ y_v = 3 \end{aligned}$$

*Therefore,*

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \det \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} = 15 + 1 = \boxed{16.}$$

**Example 6.5.4.** *Find the Jacobian of the transformation $x = u + 4v$ and $y = 3u - 2v$*
*By definition,*

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix} = \det \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = -2 - 12 = \boxed{-14}$$

**Example 6.5.5.** *Let $x = \alpha\sin\beta$ and $y = \alpha\cos\beta$ find the Jacobian of $(x,y) \mapsto (\alpha,\beta)$*
*By definition,*

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \det \begin{bmatrix} \partial x/\partial \alpha & \partial x/\partial \beta \\ \partial y/\partial \alpha & \partial y/\partial \beta \end{bmatrix} = \det \begin{bmatrix} \sin\beta & \alpha\cos\beta \\ \cos\beta & -\alpha\sin\beta \end{bmatrix} = -\alpha\sin^2\beta - \alpha\cos^2\beta = \boxed{-\alpha.}$$

The proof of the following is a simple consequence of the general chain rule. It is also very useful for certain problems where finding the inverse transformations is troublesome.

**Proposition 6.5.6.**

Given the transformation $T(u,v) = (x(u,v), y(u,v))$ is differentiable and invertible,

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1 \text{ which gives } \frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1}.$$

Now that we have a little experience with Jacobians, let us return to the problem of integration.

In terms of our new-found notation the central point of the last section reads:

$$\triangle x\,\triangle y = \frac{\partial(x,y)}{\partial(u,v)}\triangle u\,\triangle v$$

As $\triangle x\,\triangle y \to 0$ this formula above intuitively tells us that:

$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}du\,dv.$$

This is how we change the *measure* of integration in double integrals. Naturally, we also have to modify the integrand and bounds. The absolute value bars are needed as the sign of the integral arises from the integrand not $dA$ for a double integral[9]. In particular:

**Theorem 6.5.7.** *Changing variables in double integrals:*

Suppose $T : S \to R$ is a differentiable mapping that is mostly invertible (except possibly on the boundary) from TYPE I or II region S to TYPE I or II region R and suppose that f is a continuous function whose domain includes R,

$$\iint_R f(x,y)dx\,dy = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

where the $|\cdot|$ on the Jacobian are absolute value bars.

**Example 6.5.8.** *Let's apply this Theorem to Polar Coordinates, suppose f is continuous etc...*

$$\iint_R f(x,y)dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dr\,d\theta = \iint_S f(r\cos\theta, r\sin\theta)r dr\,d\theta$$

**Example 6.5.9.** *Using Ex.1.4.5 calculate the area of a circle of radius A, call it R*

$$\iint_R dx\,dy = \iint_S r dr\,d\theta = \int_0^{2\pi}\int_0^A r dr\,d\theta = \int_0^{2\pi}\frac{1}{2}A^2 d\theta = \frac{1}{2}A^2\cdot 2\pi = \boxed{\pi A^2}$$

**Remark 6.5.10.** $S = [0,2\pi]\times[0,A]$

This is considerably easier than the direct Cartesian calculation of area, although the same geometry makes both solutions work. Notice "mostly invertible" is a needed qualifier since the angle $\theta$ doubles up on $\theta = 0$ and $2\pi$ given $(x,y)$ along $\theta = 0$ should we say it corresponds to $\theta = 0$ or $\theta = 2\pi$? Fortunately a curve or two will not change double integral's result.

**Example 6.5.11.** *Evaluate the integral by performing an appropriate coordinate change,*

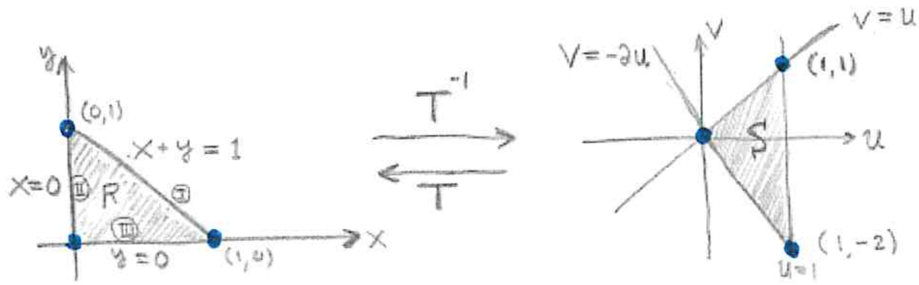
$$I \equiv \int_0^1\int_0^{1-x}\sqrt{x+y}(y-2x)^2 dy\,dx$$

*This suggests we choose* $u = x+y$ *and* $v = y-2x$. *Solving for x,y yields* $x = \frac{u}{3} - \frac{v}{3}$ *and* $y = \frac{2u}{3} + \frac{v}{3}$, *Thus*

$$T(u,v) = \left(\frac{1}{3}(u-v), \frac{1}{3}(2u+v)\right).$$

*If* $T : S \to R$ *than what is S in this case? We are interested in R that is indicated by the integral I, namely* $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1-x \;\&\; 0 \le x \le 1\}$. *The graph is justified by the analysis below the graph:*

[9] you may recall, as we calculated TYPE I or TYPE II regions the construction of $dA$ requires it be positive. This was implicit within the definitions of TYPE I and II.




*To figure out the boundaries in the uv-triangle we are guided by the knowledge that for a simple linear $T$ as we have here triangles go to triangles, vertices to vertices.*

**( I.)** $x + y = \frac{1}{3}u - \frac{1}{3}v + \frac{2}{3}u + \frac{1}{3}v = \boxed{u = 1}$

**( II.)** $0 = x = \frac{1}{3}u - \frac{1}{3}v \Rightarrow \boxed{u = v}$

**( III.)** $0 = y = \frac{2}{3}u + \frac{1}{3}v \Rightarrow \boxed{v = -2u}$.

*Thus,* $S = \{(u, v) \in \mathbb{R}^2 \mid -2u \leq v \leq u \ \& \ 0 \leq u \leq 1\}$. *Notice then, for* $x = \frac{1}{3}(u - v)$ *and* $y = \frac{1}{3}(2u + v)$ *we calculate the Jacobian:*

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{-1}{3}\right)\left(\frac{2}{3}\right) = \frac{3}{9} = \frac{1}{3}$$

*Apply what we've learned.*

$$\begin{aligned} I &= \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy\, dx \\ &= \int_0^1 \int_{-2u}^{u} \sqrt{u}\, v^2 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du \\ &= \int_0^1 \frac{1}{3} v^2 \sqrt{u}\, dv\, du \\ &= \int_0^1 \frac{\sqrt{u}}{9} v^3 \Big|_{-2u}^{u} \\ &= \int_0^1 \frac{\sqrt{u}}{9}(u^3 - (-2u)^3) du \\ &= \int_0^1 u^{3+1/2} du \\ &= \frac{2}{9} u^{9/2} \Big|_0^1 \\ &= \boxed{2/9} \end{aligned}$$

*Acknowledgment: this example borrowed from Thomas' Calculus 10th Ed, pg 1040.*