Lecture 28: Wave Equations and Complex Notation

A wave on a string with tension T and mass per unit length μ has speed $v = \sqrt{\frac{T}{\mu}}$ and if f is the distance, well coordinate, transverse to the direction of the wave propagation then*

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

describes the motion of such a wave; it's the <u>wave equation</u>. We argued last Lecture that in empty space the $\vec{E}$ & $\vec{B}$ fields solve the 3-dim'l analog of the above eq$^{\underline{n}}$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \quad \& \quad \boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$$

Apparently $\frac{1}{c^2} = \mu_0 \varepsilon_0 \Rightarrow c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ describes the speed of these electromagnetic waves. But, that is exactly the speed of light. <u>LIGHT</u> <u>IS</u> <u>AN</u> <u>ELECTROMAGNETIC</u> <u>WAVE</u>! In this lecture we focus on the mathematics of waves to better understand how to process solutions to the $\vec{E}$ & $\vec{B}$ wave eq$^{\underline{n}}$.

* **Remark:**

Generally, $\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ describes waves on some object. For instance, on a plane we'd face $f_{xx} + f_{yy} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$.

However, adopting polar coord. might be smart for certain problems like waves on a drum etc.

$$\boxed{\text{Th}^{\underline{m}}/ \quad \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \quad \text{is solved by } f(z,t) = g(z-vt)}$$

<u>Proof</u>: If g is a differentiable function then the chain-rule shows that $f(z,t) = g(z-vt)$ has

$$① \quad \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial z}\right] = \frac{\partial}{\partial z}\left[g'(z-vt)\frac{\partial}{\partial z}(z-vt)\right]$$
$$= \frac{\partial}{\partial z}\left[g'(z-vt)\right]$$
$$= g''(z-vt)\frac{\partial}{\partial z}\left[z-vt\right]$$
$$= g''(z-vt).$$

$$② \quad \frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial t}\right] = \frac{\partial}{\partial t}\left[g'(z-vt)\frac{\partial}{\partial t}\left[z-vt\right]\right]$$
$$= \frac{\partial}{\partial t}\left[g'(z-vt)(-v)\right]$$
$$= -v\, g''(z-vt)\frac{\partial}{\partial t}\left[z-vt\right]$$
$$= (-v)^2 g''(z-vt)$$
$$= v^2 g''(z-vt)$$

Therefore, by ① and ② we find $\underline{\dfrac{\partial^2 f}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}}$.

TERMINOLOGY FOR WAVE EQUATIONS

Def^n/ $$\underbrace{f(z,t) = A\cos\left(k(z-vt)+\delta\right)}_{\text{sinusoidal wave}}$$

A = amplitude, <u>assume</u> $A > 0$

$k(z-vt)+\delta$ = argument of cosine, the "phase"

δ = phase constant (beware, I drop the term "constant" at times)

customarily $0 \le \delta < 2\pi$

However, $-\pi < \delta \le \pi$ sometimes employed.

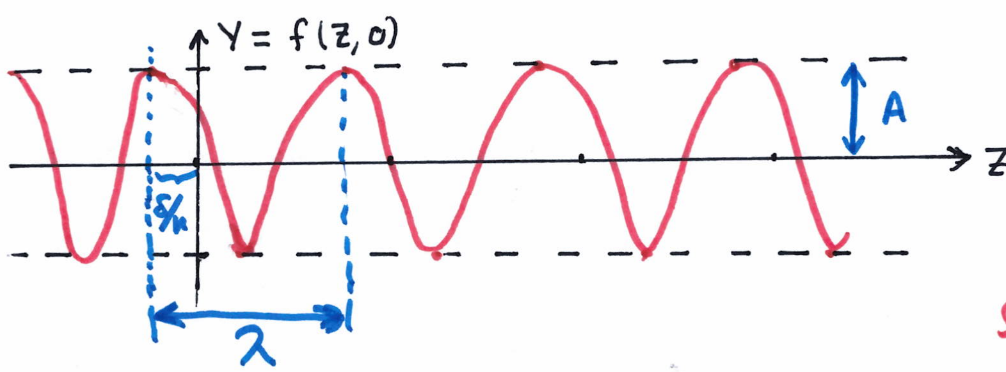
$z = vt - \delta/k$ is the central maximum

δ/k is the amount the wave is delayed.

k = wave number

$\lambda = \dfrac{2\pi}{k}$ = wavelength

$T = \dfrac{2\pi}{kv}$ = period



Sorry, sad.

Continuing the terminology for sinusoidal

$$\boxed{f(z,t) = A\cos\left(k(z - vt) + \delta\right)}$$

wave #

Speed of wave

Def^n/ $\nu = \dfrac{1}{T} = \dfrac{kv}{2\pi} = \dfrac{v}{\lambda}$

frequency (ν), period (T), wavelength (λ)

$$\omega = 2\pi\nu = kv$$

angular frequency (ω)

[**Remark**: I have used $f = \frac{1}{T} = \nu$ in other course work.]

Notice that the sinusoid can be written in terms of wave # and angular frequency most nicely,

$$① \quad \boxed{f(z,t) = A\cos(kz - \omega t + \delta)}$$

moves in $\hat{z}$ direction

assumes $v > 0$.

Likewise, a left-moving wave,

$$② \quad \boxed{f(z,t) = A\cos(\underbrace{kz + \omega t - \delta}_{\text{wave delayed by } \delta/k \text{ in its leftward motion.}})}$$

moves in $-\hat{z}$ direction

$$\downarrow$$

$$\underbrace{f(z,t) = A\cos(-kz - \omega t + \delta)}$$

same as ① but with $-k < 0$
so if we extend wave # to include negative values this picks up left-moving waves.

Complex Notation

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ ← Euler's Formula, or def$^{\underline{n}}$ of imaginary exponential.

Notice $e^{i\theta}e^{i\beta} = e^{i(\theta+\beta)}$ and $\overline{e^{i\theta}} = \cos\theta - i\sin\theta = e^{-i\theta}$

Generally, if $z = x + iy$ where $x, y \in \mathbb{R}$ then $Re(z) = x$ and $Im(z) = y$. In fact,

$$Re(z) = \frac{1}{2}(z + \bar{z})$$

$$Im(z) = \frac{1}{2i}(z - \bar{z})$$

where $\bar{z} = \overline{x+iy} = x - iy$ is the <u>complex conjugate</u>

$$\boxed{f(z,t) = A\cos(kz - \omega t + \delta) = Re\left(Ae^{i(kz-\omega t+\delta)}\right)}$$

Def$^{\underline{n}}$/ $\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$ is the <u>complex wave function</u> with <u>complex amplitude</u> $\tilde{A} = Ae^{i\delta}$

$$\boxed{f(z,t) = Re\left[\tilde{f}(z,t)\right]}$$

<u>Proof</u>: $Re(\tilde{f}(z,t)) = Re(\tilde{A}e^{i(kz-\omega t)})$

$= Re(A_o e^{i\delta} e^{i(kz-\omega t)})$

$= Re[A_o \exp(i(kz-\omega t+\delta))]$

$= A\cos(kz-\omega t+\delta)$. //

$\boxed{E1}$ Suppose we wish to add two sinusoidal waves given by $f_1 = A_1 \cos(kz - \omega t + \delta_1)$ and

$$f_2 = A_2 \cos(kz - \omega t + \delta_2)$$

then $\tilde{A}_1 = A_1 e^{i\delta_1}$ and $\tilde{A}_2 = A_2 e^{i\delta_2}$ give

$$f_1 = Re(\tilde{f}_1) \quad \text{and} \quad f_2 = Re(\tilde{f}_2)$$

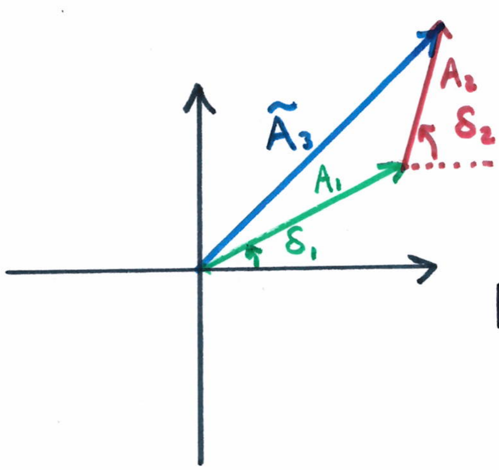
where $\tilde{f}_1 = \tilde{A}_1 e^{i(kz-\omega t)}$ and $\tilde{f}_2 = \tilde{A}_2 e^{i(kz-\omega t)}$

Observe

$$f_3 = f_1 + f_2 = Re(\tilde{f}_1) + Re(\tilde{f}_2)$$
$$= Re(\tilde{f}_1 + \tilde{f}_2)$$
$$= Re(\tilde{f}_3) \quad \text{where } \tilde{f}_3 \overset{\text{def}^{\underline{n}}}{=} \tilde{f}_1 + \tilde{f}_2$$

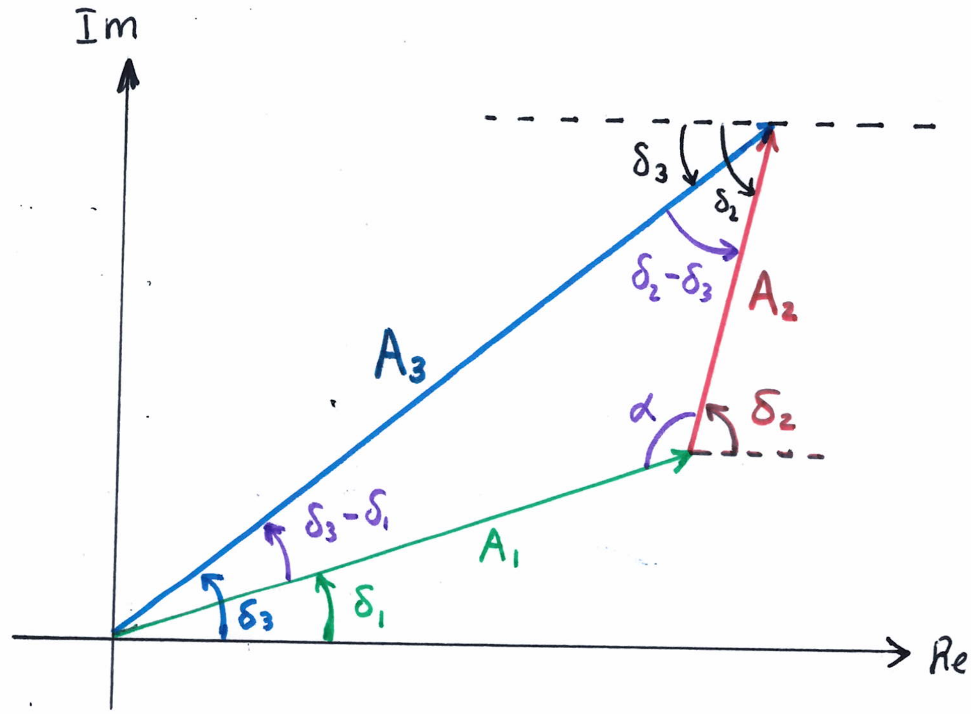
Let's work it out, let $\theta = kz - \omega t$

$$\tilde{f}_3 = \tilde{A}_1 e^{i\theta} + \tilde{A}_2 e^{i\theta}$$
$$= (A_1 e^{i\delta_1} + A_2 e^{i\delta_2}) e^{i\theta}$$



$$\tilde{A}_3 = (A_1 \cos\delta_1 + A_2 \cos\delta_2) + i(A_1 \sin\delta_1 + A_2 \sin\delta_2)$$

$$|\tilde{A}_3|^2 = (A_1 \cos\delta_1 + A_2 \cos\delta_2)^2 + (A_1 \sin\delta_1 + A_2 \sin\delta_2)^2$$
$$= A_1^2 + A_2^2 + 2A_1 A_2 (\cos\delta_1 \cos\delta_2 + \sin\delta_1 \sin\delta_2)$$
$$= A_1^2 + A_2^2 + 2A_1 A_2 \cos(\delta_1 - \delta_2)$$



$$(\delta_3 - \delta_1) + (\delta_2 - \delta_3) + \alpha = \pi$$

$$\alpha = \pi + \delta_1 - \delta_2$$

$$|\tilde{A}_3|^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(\delta_1 - \delta_2)$$

$$= A_1^2 + A_2^2 + 2A_1A_2 \cos(\alpha - \pi)$$

$$= \underbrace{A_1^2 + A_2^2 - 2A_1A_2 \cos(\alpha)}$$

Law of cosines where $\alpha = \pi + \delta_1 - \delta_2$

$$\boxed{\begin{array}{l} A_3 = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1 - \delta_2)} \\ \text{with } \delta_3 = \text{complicated... see diagram} \end{array}}$$

EXAMPLE

Let's add $f_1(z,t) = A_1 \cos(kz - \omega t)$ and the sinusoidal $f_2(z,t) = A_2 \sin(kz - \omega t)$

Notice $\sin(\theta + \pi/2) = \sin\theta \cos(\pi/2) + \cos\theta \sin(\pi/2)$

Actually, instead notice $\cos(\theta - \frac{\pi}{2}) = \cos\theta \cos\frac{\pi}{2} + \sin\theta \sin\frac{\pi}{2}$

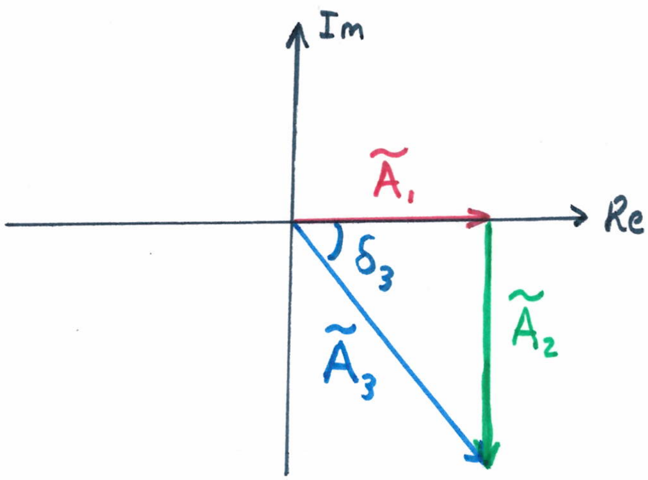
Hence, $\underline{\sin\theta = \cos(\theta - \pi/2)}$

$$f_2(z,t) = A_2 \cos(kz - \omega t - \pi/2)$$

Then $\tilde{f}_1 = A_1 e^{i\theta}$ and $\tilde{f}_2 = A_2 e^{i(\theta - \pi/2)} = A_2 e^{-i\pi/2} e^{i\theta}$

We have complex amplitudes

$$\tilde{A}_1 = A_1 \quad \& \quad \tilde{A}_2 = A_2 e^{-i\pi/2}$$



$$A_3 = \sqrt{A_1^2 + A_2^2}$$

$$\delta_3 = -\tan^{-1}\left(\frac{A_2}{A_1}\right)$$

Then $\tilde{f}_1 + \tilde{f}_2 = \tilde{f}_3 = \tilde{A}_3 e^{i\theta} = \sqrt{A_1^2 + A_2^2}\, e^{i(\theta + \delta_3)}$

$$\boxed{f_3(z,t) = \sqrt{A_1^2 + A_2^2} \cos\left(kz - \omega t - \tan^{-1}\left(\frac{A_2}{A_1}\right)\right)}$$