LECTURE 32: CRITICAL POINTS & THE HESSIAN'S SPECTRUM

[P148] Find critical points and analyze e-values of the Hessian to classify each ~~extrem~~ critical point as min/max or saddle.

(a.) $f(x,y) = 5x^2 + 8xy - 10x + 5y^2 - 8y + 5$

(b.) $f(x,y,z) = x^2 + y^2 + z^2 + 4xy + 4xz + 4yz$

(a.) $\nabla f = \langle f_x, f_y \rangle = \langle 10x + 8y - 10,\ 8x + 10y - 8 \rangle$

$$\underbrace{\nabla f = \langle 0, 0 \rangle}_{\text{critical pt. condition}} \quad \longleftrightarrow \quad \left. \begin{aligned} 10x + 8y &= 10 \\ 8x + 10y &= 8 \end{aligned} \right\} \text{ has sol}^{\underline{n}} \ (1,0).$$

Hence $(1,0)$ is only critical point. Generally,

$$[Q] = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \underbrace{\begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}}_{A}$$

Remark: generally evaluating at $(1,0)$ is more interesting.

Then,

$$\det[A - \lambda I] = \det\begin{bmatrix} 10-\lambda & 8 \\ 8 & 10-\lambda \end{bmatrix} = (\lambda - 10)^2 - 8^2$$
$$= (\lambda - 10 - 8)(\lambda - 10 + 8)$$
$$= (\lambda - 18)(\lambda - 2)$$

Hence $\lambda_1 = 18,\ \lambda_2 = 2 \Rightarrow \boxed{f(1,0) = 0 \text{ is local min.}}$

(b.) $\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x + 4y + 4z,\ 2y + 4x + 4z,\ 2z + 4x + 4y \rangle$

$$\nabla f = 0 \quad \longleftrightarrow \quad \underbrace{\begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}}_{B} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det B = 2(4-16) - 4(8-16) + 4(16-8) = -24 + 32 + 32 \neq 0$$

Thus $\underline{(0,0,0) \text{ is only critical pt.}}$

$$\text{Consider } [Q] = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ \underbrace{4 \quad 4 \quad 2}_{A} \end{bmatrix}$$

funny.

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we need to calculate the spectrum of A,

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-x & 4 & 4 \\ 4 & 2-x & 4 \\ 4 & 4 & 2-x \end{bmatrix}$$

decided to use $\lambda = x$ instead for change of pace ☺

$r_3 - r_2$

$$= \det\begin{bmatrix} 2-x & 4 & 4 \\ 4 & 2-x & 4 \\ 0 & x+2 & -2-x \end{bmatrix} \quad : \quad \text{since } 4-(2-x) = x+2 \text{ and } 2-x-4 = -(x+2)$$

$r_2 + r_3$

$$= \det\begin{bmatrix} 2-x & 4 & 4 \\ 4 & 0 & -x+2 \\ 0 & x+2 & -2-x \end{bmatrix}$$

$$= (2-x)(x+2)(-x+2)(-1) - 4(4)(-2-x) + 4(4(x+2))$$

$$= (x+2)\left[(2-x)(2+x) + 16 + 16\right]$$

$$= (x+2)\left[-x^2 + 4 + 32\right]$$

$$= (x+2)(36-x^2)$$

$$= (x+2)(6-x)(6+x) \quad \therefore \quad \underline{\lambda_1 = -2,\ \lambda_2 = 6,\ \lambda_3 = -6}.$$

Thus $\underline{f(0,0,0) = 0 \text{ is at a saddle point.}}$
(neither local min nor max)

Remark: I'm not certain my row-operations were super helpful. But, it would be wise to review to see what can help with the $\det(A-\lambda I) = 0$ calculation. Certainly row-swaps are permitted and row-add. & rescalings do not change the sol$^{\underline{n}}$ set for $\det(A-\lambda I) = 0$. Think about it...