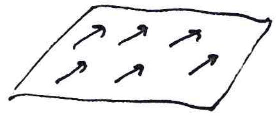
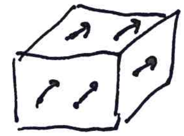
LECTURE 29: VECTOR FIELDS AND GRADIENT OPERATOR

pgs. 325–328 in 2020 notes

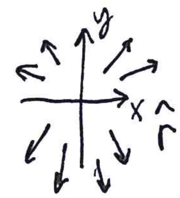
7.1 vector fields



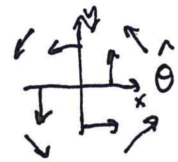


Definition 7.1.1.

> A vector field on $S \subseteq \mathbb{R}^n$ is an assignment of a n-dimensional vector to each point in S. If $\vec{F} = \langle F_1, F_2, \dots, F_n\rangle = \sum_{j=1}^{n} F_j \widehat{x}_j$ then the multivariate functions $F_j : S \to \mathbb{R}$ are called the **component functions** of $\vec{F}$. In the cases of $n = 2$ and $n = 3$ we sometimes use the popular notations
> $$\vec{F} = \langle P, Q\rangle \qquad \& \qquad \vec{F} = \langle P, Q, R\rangle.$$

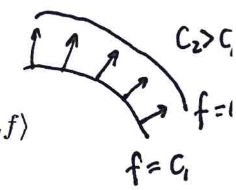


We have already encountered examples of vector fields earlier in this course.



Example 7.1.2. *Let $\vec{F} = \widehat{x}$ or $\vec{G} = \widehat{y}$. These are constant vector fields on $\mathbb{R}^2$. At each point we attach the same fixed vector; $\vec{F}(x,y) = \langle 1, 0\rangle$ or $\vec{G}(x,y) = \langle 0, 1\rangle$. In contrast, $\vec{H} = \widehat{r}$ and $\vec{I} = \widehat{\theta}$ are non-constant and technically are only defined on the punctured plane $\mathbb{R}^2 - \{(0,0)\}$. In particular,*

$$\vec{H}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle \qquad \& \qquad \vec{I}(x,y) = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle.$$



Of course, given any differentiable $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ we can create the vector field $\nabla f = \langle \partial_x f, \partial_y f\rangle$ which is normal to the level curves of f.

Example 7.1.3. *Let $\vec{F} = \widehat{x}$ or $\vec{G} = \widehat{y}$ or $\vec{H} = \widehat{z}$. These are constant vector fields on $\mathbb{R}^3$. At each point we attach the same fixed vector; $\vec{F}(x,y,z) = \langle 1,0,0\rangle$ or $\vec{G}(x,y,z) = \langle 0,1,0\rangle$ and $\vec{H}(x,y,z) = \langle 0,1,0\rangle$. In contrast, $\vec{I} = \widehat{r}$ and $\vec{J} = \widehat{\theta}$ are non-constant and technically are only defined on $\mathbb{R}^3 - \{(0,0,z) \mid z \in \mathbb{R}\}$. In particular,*

$$\vec{I}(x,y,z) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0 \right\rangle \qquad \& \qquad \vec{J}(x,y,z) = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0 \right\rangle$$

Similarly, the spherical coodinate frame $\widehat{\rho}, \widehat{\phi}, \widehat{\theta}$ are vector fields on the domain of their definition. Of course, given any differentiable $f : S \subseteq \mathbb{R}^3 \to \mathbb{R}$ we can create the vector field $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f\rangle$ which is normal to the level surfaces of f.

The **flow-lines** or **streamlines** are paths for which the velocity field matches a given vector field. Sometimes these paths which line up with the vector field are also called **integral curves**.

Definition 7.1.4.

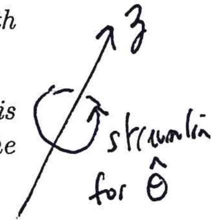
> In particular, given a vector field $\vec{F} = \langle P, Q, R\rangle$ we say $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is an **integral curve** of $\vec{F}$ iff
> $$\vec{F}(\vec{r}(t)) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \qquad a.k.a \qquad \frac{dx}{dt} = P, \ \frac{dy}{dt} = Q, \ \frac{dz}{dt} = R.$$

We need to integrate each component of the vector field to find this curve. Of course, given that P, Q, R are typically functions of x, y, z the "integration" requires thought. Even in differential equations(334) the general problem of finding integral curves for vector fields is beyond our standard techniques for all but a handful of well-behaved vector fields. That said, the streamlines for the examples below are geometrically obvious so we can reasonably omit the integration.

$\vec{F} = \langle 1, 0, 0 \rangle$

$\frac{dx}{dt} = 1, \quad \frac{dy}{dt}, \quad \frac{dz}{dt} = 0$

Example 7.1.5. *Suppose $\vec{F} = \widehat{x}$ then obviously $x(t) = x_o + t, y = y_o, z = z_o$ is the streamline of $\widehat{x}$ through (x_o, y_o, z_o). Likewise, I think you can calculate the steamlines for $\widehat{y}$ and $\widehat{z}$ without much trouble. In fact, any constant vector field $\vec{F} = \vec{v}_o$ simply has streamlines which are lines with direction vector $\vec{v}_o$.*

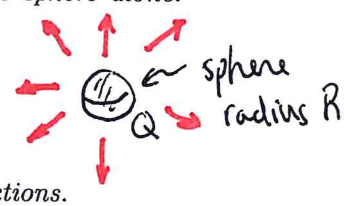


Example 7.1.6. *The magnetic field around a long steady current in the postive z-direction is conveniently written as $\vec{B}(r, \theta, z) = \frac{\mu_o I}{2\pi r}\widehat{\theta}$. The streamlines are circles which are centered on the axis and point in the $\widehat{\theta}$ direction.*

Example 7.1.7. *If a charge Q is distributed uniformly through a sphere of radius R then the electric field can be show to me a function of the distance from the center of the sphere alone. Placing that center at the origin gives*

$k = \frac{Q}{4\pi\epsilon_0}$ (not consistent with typical physics notation)

$$\vec{E}(\rho) = \widehat{\rho}\begin{cases} \frac{k\rho}{R^3} & 0 \leq \rho \leq R \\ \frac{k}{\rho^2} & \rho \geq R \end{cases}$$



The streamlines are simply lines which flow radially out from the origin in all directions.

Challenge: in electrostatics the density of streamlines (often called fieldlines in physics) is used to measure the magnitude of the electric field. Why is that reasonable?

Example 7.1.8. *The other side of the thinking here is that given a differential equation we could use the plot of the vector field to indicate the flow of solutions. We can solve numerically by playing a game of directed connect the dots which is the multivariate analog of Euler's method for solving $dy/dx = f(x, y)$.*

pplane

$$\frac{dx}{dt} = 2x(y^2 + z^2), \qquad \frac{dy}{dt} = 2x^2 y, \qquad \frac{dz}{dt} = 2x^2 z$$

We'd look to match the curve up with the vector-field plot of $\vec{F} = \langle 2x(y^2 + z^2), 2x^2 y, 2x^2 z \rangle$. This particular field is a gradient field with $\vec{F} = \nabla f$ for $f(x, y, z) = x^2(y^2 + z^2)$. Solutions to the differential equations describe paths which are orthogonal to the level surfaces of f since the paths are parallel to ∇f.

Perhaps you can see how this way of thinking might be productive towards analyzing otherwise intractable problems in differential equations. I merely illustrate here to give a bit more breadth to the concept of a vector field. Of course Stewart[3] has pretty pictures with real world jutsu so you should read that if this is not *real* to you without those comments.

[3] we are covering chapter 17 from here on out