LECTURE 2: SPACETIME

Technically what I call <u>Minkowski</u> <u>Space</u> is just one model of such a space. If such a space exists in nature then it will be in bijective correspondance with $\mathcal{M}$ as defined below. (this amounts to the existence of an inertial observer) Carrol describes how to set-up such in §1.2

Def^n/ $\mathcal{M} = \mathbb{R}^4 = \{(x^0, x^1, x^2, x^3) \mid x^0, x^1, x^2, x^3 \in \mathbb{R}\}$ where points in $\mathcal{M}$ are known as <u>events</u>. The <u>interval</u> between two events (x_1^μ) and (x_2^μ) is given by the <u>Minkowski</u> <u>metric</u> acting on $\Delta x^\mu = x_2^\mu - x_1^\mu$

$$(\Delta x^\mu) = (x_2^0 - x_1^0,\ x_2^1 - x_1^1,\ x_2^2 - x_1^2,\ x_2^3 - x_1^3) = \Delta x$$

Matrix of Minkowski Metric

$$\begin{aligned}(\Delta s)^2 = \underset{\substack{\uparrow\\ \text{Minkowski}\\ \text{Metric}}}{g}(\Delta x, \Delta x) &= (\Delta x)^T \eta\, \Delta x \\ &= [\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3] \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{bmatrix} \\ &= \underbrace{-(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}_{\text{invariant interval between events 1 and 2.}}\end{aligned}$$

<u>NOTATION</u>:

$$x^\mu: \quad \begin{aligned} x^0 &= ct \leftarrow \text{time} \\ \left.\begin{aligned} x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}\right\} &\ \text{space} \end{aligned}$$

maybe $\mathcal{M}$ should be called <u>timespace</u>.

The increment between events Ⓐ and Ⓑ we can denote by ΔX

$$\Delta X = (A^0 - B^0,\ A^1 - B^1,\ A^2 - B^2,\ A^3 - B^3)$$

This is a particular instance of a <u>4-vector</u>

Def^{n}/ A 4-vector $V = (V^0, V^1, V^2, V^3) = (V^0, \vec{V}) = (V^0, V^i)$ then for 4-vectors V, W we <u>define</u> the <u>Minkowski</u> <u>Metric</u> by

$$g(V,W) = -V^0W^0 + V^1W^1 + V^2W^2 + V^3W^3$$
$$= -V^0W^0 + \vec{V}\cdot\vec{W}$$

The Minkowski Metric restricts to the usual dot-product $\vec{V}\cdot\vec{W}$ on the <u>spatial</u> parts of V & W. However, strictly speaking g is <u>not</u> a metric or inner-product since $g(V,V) = 0 \not\Rightarrow V = 0$

$\boxed{E1}$ $V = \langle 1,1,0,0\rangle$

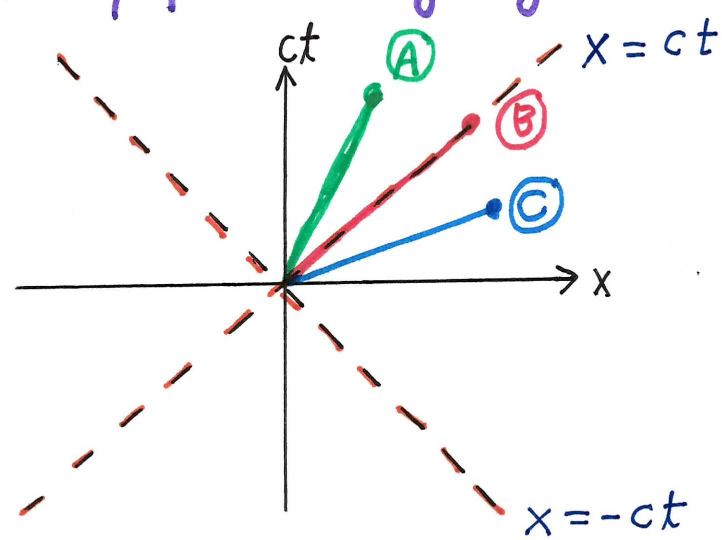
$$g(V,V) = [1,1,0,0]\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = [1,1,0,0]\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = -1+1 = 0.$$

Likewise $\langle 1,0,1,0\rangle$, $\langle 1,0,0,1\rangle$ are <u>null vectors</u>.

<u>Remark</u>: the null vectors above all correspond to increments formed by following the path of light from one event to another, they fall on a <u>light cone</u>.

SPACETIME DIAGRAMS

- suppress y, z and consider only $x^0 = ct$ and $x^1 = x$
- study particle going from origin to...



A : in future $(\Delta S_A)^2 < 0$ (TIMELIKE)

B : on lightcone $(\Delta S_B)^2 = 0$ (LIGHTLIKE)

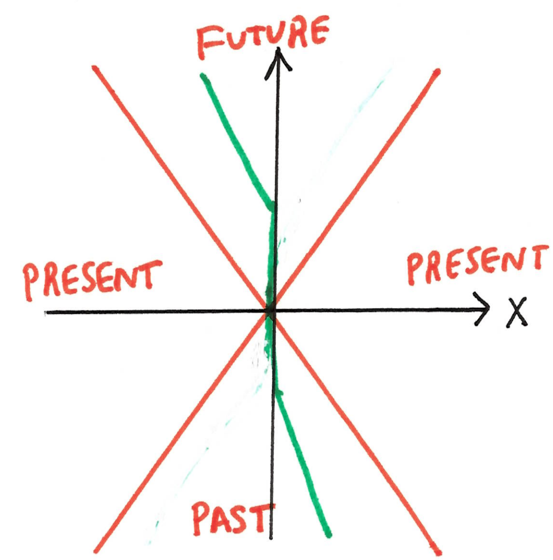
C : in present $(\Delta S_B)^2 > 0$ (SPACELIKE)

A : possible motion for massive object

B : possible motion for light

C : impossible motion for physical object

- generally objects follow paths known as <u>worldlines</u>. For a massive object the increment between events on a given world line should be timelike



Def^{n}/ Proper time between events given by $(\Delta\tau)^2 = -(\Delta s)^2$

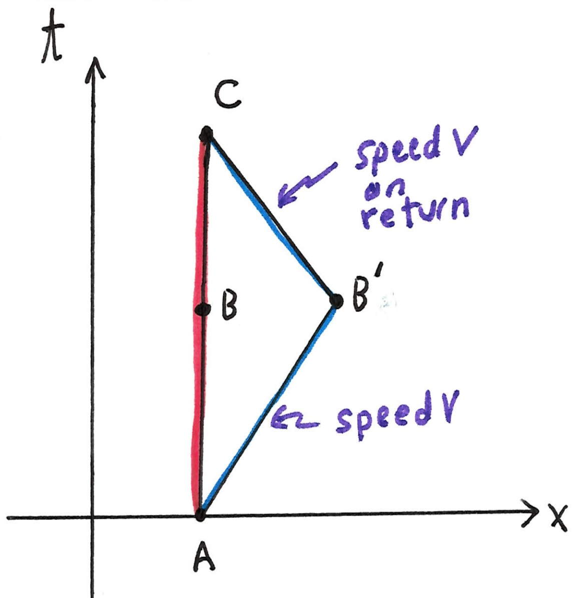
For a path through spacetime (aka a worldline)

$$\Delta s = \int_{C_1} \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda \quad \text{where } C_1 \text{ is spacelike path}$$

$$\Delta \tau = \int_{C_2} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda \quad \text{where } C_2 \text{ is timelike path}$$

Infinitesimally, the interval is $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ whereas the square of the proper time $d\tau^2 = -ds^2$

Remark: the discussion below uses $c = 1$ convention



A: $(0, 0)$

B: $(\Delta t/2, 0)$

C: $(\Delta t, 0)$

B': $\left(\frac{\Delta t}{2}, \Delta x\right)$

$\Delta x = \frac{1}{2} v \Delta t$

$$(\Delta\tau)^2 = -(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

$$\Delta\tau_{ABC} = \Delta t$$

$$\Delta\tau_{AB'} = \sqrt{(\Delta t/2)^2 - \left(\frac{v\Delta t}{2}\right)^2} = \sqrt{\frac{1}{4} - \frac{v^2}{4}}\, \Delta t = \frac{1}{2}\sqrt{1 - v^2}\, \Delta t = \Delta\tau_{B'C}$$

- Comparing proper times of stationary twin vs. moving twin gives us the TWIN PARADOX

$$\boxed{\underbrace{\Delta\tau_{AB'C}}_{\text{traveller time}} = \sqrt{1 - v^2}\, \underbrace{\Delta t}_{\text{stationary time}}}$$