LECTURE 2

1 Relations

A relation is simply some subset of a Cartesian product. This serves to generalize a function. In fact, a function is a special type of relation.

Many of the constructions we have experience with for functions will also make sense for relations. For example, we can think about the graph of a relation, the composite of two relations and the inverse of a relation. We also study how equivalence relations generalize the idea of equality and order relations generalize the concept of inequality. Equivalence relations are particularly important since their equivalence classes can be used to form a natural partition. In fact, equivalence relations are central to much of modern abstract mathematics.

Let us begin with the definition:

$x \not R y$

Definition 1.1. *Let A, B be sets. We say that R is a relation from A to B if $R \subseteq A \times B$. Moreover, we say that xRy if $(x,y) \in R$. If xRy then we say that x is related to y. On the other hand if $(x,y) \notin R$ then we say that x is not related to y and write $x \not R y$. When we say xRy, I will call x the input of the relation and y the output of R.*

$$Domain(R) = \{x \in A \mid y \in B \text{ such that } xRy\}$$

$$Range(R) = \{y \in B \mid x \in A \text{ such that } xRy\}$$

Finally, if $R \subseteq A \times A$ and $dom(R) = A$ then we say R is a relation on A.

Notice that a relation can have more than one output for a given input. This means there are relations which cannot be thought of as a function[1]. Let me begin with a silly example:

Example 1.2. *(people) Let S be the set of all living creatures on earth. We can say that x is R-related to y if both x and y are people. In this sense I am R-related to Trump. In contrast, I am not R-related to Napolean because he's dead. I am also not R-related to my mom's dogs. They may be treated like humans but the fact remains they have tails and other dog parts that necessarily disqualify them from the category of people.*

Another silly example:

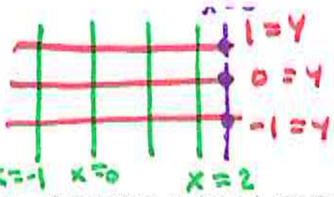
Example 1.3. *(unrelated relation) Let S be the set of all living creatures on earth. We say that x is NR-related to y if x is not the direct decendent of y. In this sense I am NR-related to Trump. In contrast, my daughter Hannah is not NR-related to me since she is my direct decendent.*

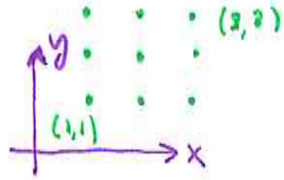
Whenever we have a relation from $\mathbb{R}$ to $\mathbb{R}$ we can picture the relation in the Cartesian plane. (We can also do the same for relations from $\mathbb{N}$ to $\mathbb{N}$ and other subsets of the real numbers)

Example 1.4. *(circle) Define $R = \{(x,y) \mid x^2 + y^2 = 1\}$. This is a relation from $\mathbb{R}$ to $\mathbb{R}$. The graph(R) is clearly a circle.*

Example 1.5. *(disk) Define $R = \{(x,y) \mid x^2 + y^2 \leq 1\}$. This is a relation from $\mathbb{R}$ to $\mathbb{R}$. The graph(R) is clearly a circle shaded in; that is the graph is a disk.*

[1] Recall that functions have one output for a given input; that is, functions must be single-valued





Example 1.6. *(positive lattice) Define $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} |\ x, y \in \mathbb{N}\}$. This is a relation from $\mathbb{R}$ to $\mathbb{R}$. The graph(R) is a grid of points. Notice that is is not a relation on $\mathbb{R}$ since dom$(R) = \mathbb{N}$.*

Example 1.7. *(integer coordinate grid) Define $R = \{(x, y) \mid x \in \mathbb{Z}, y \in \mathbb{R}\} \cup \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{Z}\}$. This is a relation from $\mathbb{R}$ to $\mathbb{R}$. The graph(R) is a grid of horizontal and vertical lines.*

There is no end to these geometric examples. Let me give a weirder example:

Example 1.8. *(rational numbers) Define $R = \{(x, y) \mid x, y \in \mathbb{Q}\}$. This is a relation from $\mathbb{R}$ to $\mathbb{R}$. For example, $3/4\ R\ 13/732$. However, π is not related to anything since $\pi \notin \mathbb{Q}$. This means that points in the xy-plane with x-coordinate π will not be included in the graph of R. However, points with $x = 3.1415 = 31415/1000$ will be included in the graph so it is hard to see the holes along $x = \pi$. In fact, the graph(R) looks like the whole plane. However, it has holes infinitely close to any point you pick. This is a consequence of the fact that there are infinitely many irrational numbers between any two distinct rational numbers (we discuss this further later in this course)*

2 composite relations

Definition 2.1. *(composite relation) Let R be a relation from A to B and let S be a relation from B to C. The composite of R and S is*

$$S \circ R = \{(a, c) \mid \text{there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$$

Example 2.2. *Let $R = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$ this is a relation from $A = \{1, 3, 5, 7\}$ to $B = \{2, 4, 6, 8\}$. Define $S = \{(2, 1), (4, 3), (5, 4), (7, 6)\}$. We see that S is a relation from B to A. Notice that $S \circ R$ is a relation from A to A; $S \circ R : A \to B \to A$:*

$S = \{(2,1), (4,3), (6,5), (8,7)\} \subseteq B \times A$

$$S \circ R = \{(1, 1), (3, 3), (5, 5), (7, 7)\}$$

$\underset{R}{(1,2)}\ \underset{S}{(2,1)}$

Since $1R2$ and $2S1$ we have $1 S \circ R 1$ and so forth... Likewise we can verify that $R \circ S$ is a relation from B to B; $R \circ S : B \to A \to B$:

$R \circ S : B \xrightarrow[S]{} A \xrightarrow[R]{} B$

$$R \circ S = \{(2, 2), (4, 4), (6, 6), (8, 8)\}$$

The relations we just exhibited are known as the **identity relations** *on A and B respective. We denote $I_A = S \circ R$ and $I_B = R \circ S$. The relations given in this example are inverses of each other.*

Definition 2.3. *(inverse relation) Given a relation R from A to B we define the inverse relation R^{-1} to be the relation from B to A defined by $R^{-1} = \{(y, x) \mid (x, y) \in R\}$*

Theorem 2.4. *The inverse relation of a relation is a relation. Moreover, domain(R^{-1}) = range(R) and range(R^{-1}) = domain(R).*

Proof: immediate from definition of R^{-1} $\square$.

The concept of an inverse relation is nice in that it avoids some of the rather cumbersome restrictions that come with the idea of an inverse function. We'll get into those restrictions soon, but you probably recall from precalculus courses that in order for the inverse function to exist we need the function's graph satisfy the horizontal line test. Inverse relations have no such restriction.

Reflexive : R is reflexive iff xRx for all $x \in X$

Symmetric : R is symmetric iff forall $x, y \in X$, xRy implies yRx

Anti-Symmetric: R is *anti-symmetric* iff for all $x, y \in X$, if $x\,R\,y$ and $y\,R\,x$, then $x = y$.

Transitive: R is *transitive* iff for all $x, y, z \in X$, $x\,R\,y$ and $y\,R\,z$ implies $x\,R\,z$.

Here are a few special types of relations:

Equivalence Relation: R is an *equivalence relation* iff it is *reflexive*, *symmetric*, and *transitive*.

Partial Order: X is *partially ordered* by R (or R is a *partial order* on X) iff R is *reflexive*, *anti-symmetric*, and *transitive*.

Total/Simple/Linear Order: X is *totally ordered* by R (or *simply ordered* or *linearly ordered*) iff R is a *partial order* and in addition for each $x, y \in X$ we have $x\,R\,y$ or $y\,R\,x$.

Let me note that partial and total orders have many variant definitions. These differences are either superficial and in the end, logically equivalent to our definition, or sometimes alternate definitions capture orderings more like "$<$" rather than "$\leq$". In such a case, when "$x\,R\,y$" is replaced with "$x\,R\,y$ or $x = y$" our notions of partial order and total order are recovered.

Notice that regular old equality (on some fixed set) is an equivalence relation[2].

Example 3.2. *(equality) Suppose that $S = \mathbb{R}$. Let $x, y \in S$, define xRy iff $x = y$. Observe that*

$$\underbrace{x = x}_{\text{reflexive}}, \quad \underbrace{x = y \Rightarrow x = y}_{\text{symmetric}}, \quad \underbrace{x = y \text{ and } y = z \Rightarrow x = z}_{\text{transitive}}$$

Therefore, R is reflexive, symmetric and transitive. Hence equality is an equivalence relation on $\mathbb{R}$.

We will introduce more interesting equivalence relations below. Next, $\leq$ on the set of real numbers $\mathbb{R}$ is a total ordering (thus also a partial ordering).

Example 3.3. *(total order relation) Suppose that $S = \mathbb{R}$. For $x, y \in \mathbb{R}$, define xRy iff $x \leq y$. Since $x \leq x$ for all $x \in \mathbb{R}$ we see R is reflexive. If $x \leq y$ and $y \leq x$ then $x = y$ hence R is antisymmetric. Transitivity is also clear since $x \leq y$ and $y \leq z$ implies $x \leq z$. Finally any pair of real numbers $x, y \in \mathbb{R}$ has either $x \leq y$ or $y \leq x$ hence $\leq$ is a* **total ordering**. *This is not surprising as the abstract definition of total ordering was designed precisely to abstract $\leq$.*

Finally, given a set X, $\mathcal{P}(X)$ (the power set of X) is partially ordered by $\subseteq$ (inclusion). Note that this is *not* a total order when X has at least 2 elements since in this case we can finds subsets of X, A and B, such that $A \not\subseteq B$ and $B \not\subseteq A$.

Example 3.4. *Let X be a non-empty set and define $R = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid A \cap B = \emptyset\}$. So in other words, $A\,R\,B$ iff A and B are disjoint subsets of X. This relation is not reflexive: $X \not{R}\, X$ since $X \cap X = X \neq \emptyset$. This relation is symmetric since A and B are disjoint if and only if B and A are disjoint. This relation fails to be anti-symmetric since just because A and B are disjoint does not mean that $A = B$. Also, this relation fails to be transitive since if A and B are disjoint as well as B and C are disjoint, then it does not follow that A and C are disjoint (consider $A = C$).*

Now let us turn our attention more fully to equivalence relations.

[2] The choice of $S = \mathbb{R}$ could be modified and the calculations in the above example would still work

$(a,b)\,R\,(c,d) \longrightarrow (a,b) \sim (c,d)$

$(a,b) = \frac{a}{b}$ | $\frac{a}{b} = \frac{c}{d}$
$(c,d) = \frac{c}{d}$ | $ad = bc$

Example 3.5. *(Rational Numbers) $X = \mathbb{Z} \times \mathbb{Z}_{\neq 0}$. Let $(a,b), (c,d) \in X$. Define $(a,b) \sim (c,d)$ iff $ad = bc$.*

- *$(a,b) \sim (a,b)$ since $ab = ba$. Therefore, $\sim$ is reflexive.*
- *If $(a,b) \sim (c,d)$, then $ad = bc$. Thus $cb = da$ so $(c,d) \sim (a,b)$. Thus $\sim$ is symmetric.*
- *Suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $ad = bc$ and $cf = de$. Multiplying the first equation by f and the second equation by b, we get that $adf = bcf$ and $bcf = bde$. Thus $afd = bed$. Now recall that $d \neq 0$ (since $(c,d) \in X = \mathbb{Z} \times \mathbb{Z}_{\neq 0}$) so $af = be$. Thus $(a,b) \sim (e,f)$. Therefore, $\sim$ is transitive.*

We have just proved that $\sim$ is an equivalence relation. This really isn't that surprising considering that $\frac{a}{b} = \frac{c}{d}$ iff $ad = bc$. Our relation is merely encoding equality of fractions. No wonder so many elementary and middle school students have troubles with fractions. Equivalence of fractions is many students' first exposure to a non-trivial mathematical equivalence relation.

Definition 3.6. *Suppose that $\sim$ is an equivalence relation on X. For each $a \in X$, let $[a] = \{x \in X \mid x \sim a\}$. Thus $[a]$ is the set of all the elements of X which are related to a. We call $[a]$ the equivalence class of a, and we say that a is a representative of this equivalence class. The set of all equivalence classes is often denoted $X/\sim$.*

At this point it is worth mentioning that there is no *standard* notation for equivalence classes. Another choice which might be convenient for equivalence relation R on X is that x/R denotes the equivalence class containing $x \in X$ and $X/R = \{x/R \mid x \in X\}$.

Definition 3.7. *Let $\mathcal{P} \subseteq \mathcal{P}(X)$ and suppose that $\emptyset \notin \mathcal{P}$ ($\mathcal{P}$ is a collection of non-empty subsets of X). Next, suppose that $\cup\mathcal{P} = X$. This means that for each $x \in X$ there exists some $A \in \mathcal{P}$ such that $x \in A$. Finally, suppose that given $A, B \in \mathcal{P}$, either $A \cap B = \emptyset$ or $A = B$. This means that distinct elements of $\mathcal{P}$ are disjoint. In such a case, we call $\mathcal{P}$ a partition of X.*

Theorem 3.8. *Let $\sim$ be an equivalence relation on X. Then the equivalence classes of $\sim$ partition X. Conversely, given a partition $\mathcal{P}$ of X, define $a \sim b$ iff there exists some $E \in \mathcal{P}$ such that $a, b \in E$. Then $\sim$ is an equivalence relation on X whose equivalence classes are precisely the elements of $\mathcal{P}$.*

Proof: Let $\sim$ be an equivalence relation on X. Let $a \in X$. Then $\sim$ is reflexive so $a \sim a$. Thus $a \in [a]$. This means that every equivalence class is non-empty. Also, this shows that every element of X belongs to some equivalence class. Therefore to establish that the equivalence classes of $\sim$ partition X it only remains to show that distinct equivalence classes are disjoint.

Suppose $a, b \in X$ and $[a] \cap [b] \neq \emptyset$. We must show that $[a] = [b]$. Note that since $[a] \cap [b] \neq \emptyset$, there exists some $c \in [a] \cap [b]$. Thus $c \sim a$ and $c \sim b$. Our relation is symmetric so we also have $a \sim c$. Then since $a \sim c$ and $c \sim b$ by transitivity we have $a \sim b$. Again by symmetry we have $b \sim a$.

Suppose that $x \in [a]$. Then, by definition, $x \sim a$. So since $x \sim a$ and $a \sim b$, by transitivity we have $x \sim b$. This means $x \in [b]$ and so $[a] \subseteq [b]$. Likewise, suppose $x \in [b]$. Then $x \sim b$ and $b \sim a$ so $x \sim a$. Thus $x \in [a]$ and so $[b] \subseteq [a]$. Therefore, $[a] = [b]$.

Conversely, suppose $\mathcal{P}$ is a partition of X. Define $a \sim b$ iff there exists some $E \in \mathcal{P}$ such that $a, b \in E$.

Example 2.5. *Let $S = \{(x, \sin(x)) \mid x \in \mathbb{R}\}$. This is a relation from $\mathbb{R}$ to $\mathbb{R}$. We can visualize S as the graph of the sine function in the xy-plane. The inverse of S is*

$$S^{-1} = \{(sin(y), y) \mid y \in \mathbb{R}\}.$$

Consider that S^{-1} should be the same as the graph of the sine function except that $x = \sin(y)$ instead of $y = \sin(x)$. If you think about this for a moment or two you'll see that the graph of S^{-1} is the same as the graph of S just instead of running along the x-axis it runs up the y-axis.

Theorem 2.6. *Let A, B, C, D be sets. Suppose R, S, T are relations with $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq C \times D$*

(a.) $(R^{-1})^{-1} = R$

(b.) $T \circ (S \circ R) = (T \circ S) \circ R$

(c.) $I_B \circ R = R$ *and* $R \circ I_A = R$

(d.) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Let $R \subseteq A \times B$ then,
$(x,y) \in R \Leftrightarrow (y,x) \in R^{-1}$ by def^n of R^{-1}
$\Leftrightarrow (x,y) \in (R^{-1})^{-1}$ by def^n of inverse applied to R^{-1}
$\therefore R = (R^{-1})^{-1}$.

Proof: likely homework or example in my presentation of this topic $\square$.

Item (*b.*) says that we can write $T \circ S \circ R$ without ambiguity, composition of relations is associative. Item (*d.*) is sometimes called the "*socks-shoes principle*". Think of it this way, if I put my socks on first and then second my shoes then when I take off my socks and shoes I have to take off my shoes first and then my socks.

3 relations on a set

AND domain$(R) = X$

Let me restate the definition of a relation on a set here once more for our convenience.

Definition 3.1. *Let X be a set. If $R \subseteq X \times X$, then R is said to be a relation on X. Instead of writing $(a, b) \in R$, we will write $a\, R\, b$ or if $(a, b) \notin R$, we will write $a \not R\, b$*

Relations abound in mathematics and in regular life too. We could speak of relations on the set of people like "A is a brother of B" or "A is B's aunt" or "A and B are neighbors". In mathematics, we have relations on sets of numbers like "$\leq$", "$>$", and "sum to a rational number". Another familiar relation is that of "$\subseteq$" when dealing with sets.

It is quite useful to abstract the concept of equality. Relations which behave like "equals" are called "equivalence relations" (which are defined below). Another important kind of relation abstracts the properties of $\leq$ and $\subseteq$. We call such relations "partial orders". Let us give names to some familiar properties.

Let R be a relation on a set X.

Reflexive: R is *reflexive* iff $x\, R\, x$ for all $x \in X$.

Symmetric: R is *symmetric* iff for all $x, y \in X$, $x\, R\, y$ implies $y\, R\, x$.

First, let $a \in X$. Then since $\mathcal{P}$ is a partition, there exists some $A \in \mathcal{P}$ such that $a \in A$. Thus $a, a \in A$ so $a \sim a$ (our relation is reflexive). Next, suppose $a \sim b$. Then there exists some $E \in \mathcal{P}$ such that $a, b \in E$ so $b, a \in E$ thus $b \sim a$ (our relation is symmetric). Finally, suppose $a \sim b$ and $b \sim c$. Therefore, there exists some $E, E' \in \mathcal{P}$ such that $a, b \in E$ and $b, c \in E'$. Thus $b \in E \cap E'$ so that $E \cap E' \neq \emptyset$. Now distinct sets in a partition are disjoint. Thus $E = E'$ so $a, b, c \in E = E'$. In particular $a, c \in E$. Thus $a \sim c$ (our relation is transitive).

Let $E \in \mathcal{P}$. Then $E \neq \emptyset$ so there exists some $a \in E$. Notice that $x \in E$ implies $a, x \in E$ which implies $x \sim a$. Thus $x \in [a]$, so $E \subseteq [a]$. Suppose that $x \in [a]$. Then $x \sim a$ so there exists some $E' \in \mathcal{P}$ such that $a, x \in E'$. But $a \in E \cap E'$ so $E = E'$. Therefore, $a, x \in E = E'$. In particular, $x \in E$. Thus $[a] \subseteq E$ and so $E = [a]$. We have now shown that the equivalence classes of $\sim$ are the same as the elements of $\mathcal{P}$. $\square$

So every equivalence relation yields a partition and every partition yields an equivalence relation. Now we can use these concepts interchangeably.

Example 3.9. *Suppose that $S = \mathbb{Z}$. Suppose $x, y \in \mathbb{Z}$, define xRy iff $x - y$ is even. This is a fancy way of saying that even integers are related to even integers and odd integers are related to odd integers. Clearly R is reflexive since $x - x = 0$ which is even. Let $x, y \in \mathbb{Z}$ and assume xRy thus $x - y = 2k$ for some $k \in \mathbb{Z}$. Observe $y - x = -2k = 2(-k)$ hence yRx which shows R is symmetric. Finally, suppose $x, y, z \in \mathbb{Z}$ such that xRy and yRz. This means there exist $m, k \in \mathbb{Z}$ such that $x - y = 2k$ and $y - z = 2m$. Consider,*

$$x - z = x - y + y - z = 2k + 2m = 2(k + m)$$

Hence $x - z$ is even and we have shown xRz so R is transitive. In total we conclude R is an equivalence relation on $\mathbb{Z}$. Since each integer is either even or odd we have just two equivalence classes which partition $\mathbb{Z}$; $0/R = 2\mathbb{Z}$ and $1/R = 1 + 2\mathbb{Z}$.

The notation $k + n\mathbb{Z} = \{k + nz \mid z \in \mathbb{Z}\}$. In fact, if you take a course in abstract algebra or number theory you will likely learn a generalization of the above equivalence relation where we suppose integer x is related to y provided $y - x$ is a multiple of n. That can be shown to be an equivalence relation and the set of equivalence classes denoted $\mathbb{Z}/n\mathbb{Z}$ are known as the **modular integers**. These have been used since the time of Gauss to unravel difficult questions in number theory. I don't expect we need them in this course so I merely mention this without proof.