LECTURE 33: JEFIMENKO'S EQUATIONS & LIENARD-WIECHERT POTENTIALS

We mentioned last class that plugging retarded time $t_r = t - \mathscr{r}/c$ into Coulomb's Law and Biot-Savart did not produce correct $\vec{E}, \vec{B}$ fields for time variate sources. What is correct modification? These we can derive from the retarded potentials

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$$

We need to calculate $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$

We already calculated $-\nabla V$ last lecture (see 6)

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi}\int \frac{\partial}{\partial t}\left[\frac{1}{\mathscr{r}}\vec{J}(\vec{r}',t_r)\right]d\tau' = \frac{\mu_0}{4\pi}\int \frac{\dot{\vec{J}}}{\mathscr{r}}\,d\tau'$$

Consequently, using our formula for ∇V from Lecture 32,

$$\boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c^2\mathscr{r}}\right]d\tau'}$$

further calculation similar to those of last lecture yield from $\vec{B} = \nabla \times \vec{A}$ that,

$$\boxed{\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}^2} - \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c\mathscr{r}}\right]\times\hat{\mathscr{r}}\,d\tau'}$$

Remark: the boxed eq's above are Jefimenko's Equations. Problem 10.14 shows a rather striking application of these towards the task of justifying the quasistatic approx.

Liénard-Wiechert Potentials

Our formulation of the retarded potentials was for a continuous distribution of charge & current. Now we turn to the problem of the point charge. What are the $\vec{E}$ and $\vec{B}$ fields due to a moving charge?

SET-UP: Let $\vec{w}(t)$ be the position of point charge q at time t

Naively, we might expect

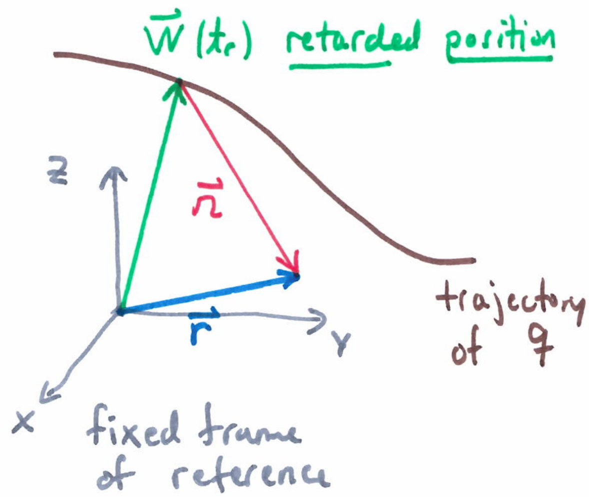
$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \implies V(\vec{r},t) = \frac{q}{\mathscr{r}}$$

However this is flawed because $\int \rho(\vec{r}\,',t_r)\,d\tau' \neq q$ for subtle reasons. If q moves around then that integral need not give q. (GRIFFITH'S GIVES PAGE LONG HEURISTIC JUSTIFICATION, p. 457-458)

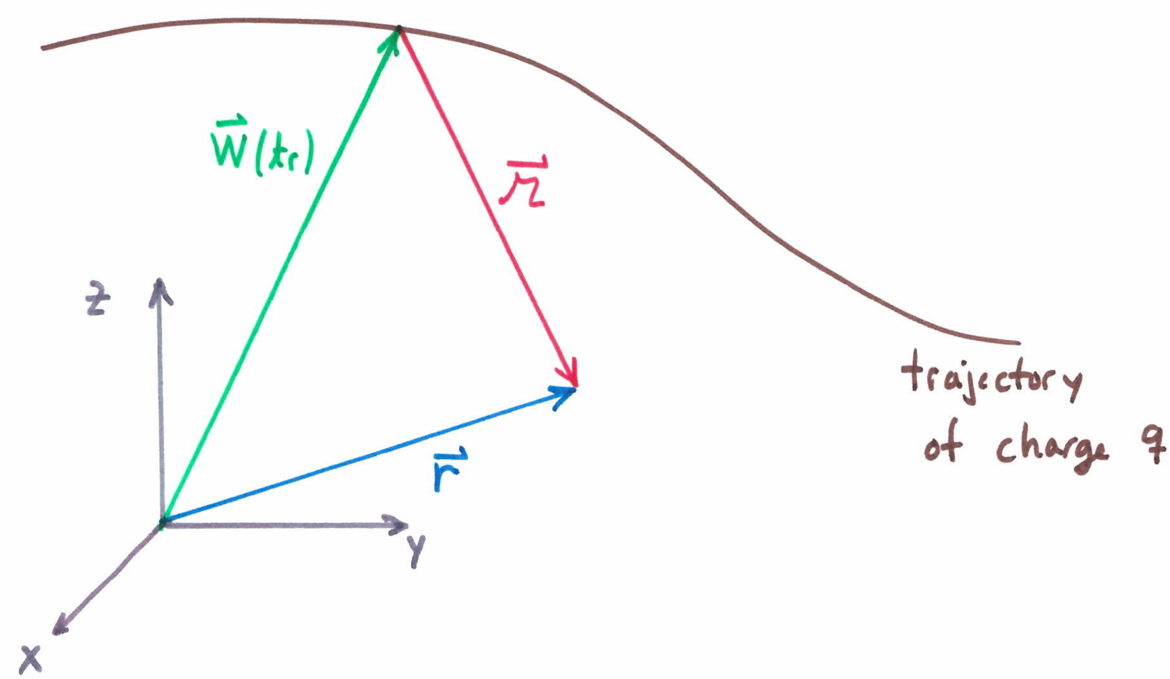
$$\tau' = \frac{\tau}{1-\hat{\mathscr{r}}\cdot\vec{v}/c} \quad \text{where} \quad \underbrace{\vec{v} = \frac{d\vec{w}}{dt}(t_r)}_{\text{velocity of } q \text{ at retarded time}}$$

Retarded time defined by

$$\underbrace{\|\vec{r}-\vec{w}(t_r)\|}_{\text{distance for news to travel}} = c\underbrace{(t-t_r)}_{\text{time to make the trip at light speed}}$$



$$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$$



Remark: at most one point on trajectory of q is in "communication" with the field point $\vec{r}$ at a given time

$$\mathscr{r}_1 = c(t - t_1) = c(t - t_2) = \mathscr{r}_2 \Rightarrow \frac{\mathscr{r}_2 - \mathscr{r}_1}{t_2 - t_1} = c$$

Where $\vec{\mathscr{r}}_1 = \vec{r} - w(t_1)$ and $\vec{\mathscr{r}}_2 = \vec{r} - w(t_2)$

$\Rightarrow$ average velocity w.r.t. $\vec{r}$ of charge is c.
(impossible hence the remark)

$$\vec{V} = \frac{d\vec{W}}{dt}(t_r)$$

Consequently,

$$\boxed{V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{V})}}$$

Then since the current density for point charge is $\rho\vec{V}$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\rho(\vec{r}', t_r)\vec{V}(t_r)}{\mathscr{r}} d\tau' = \frac{\mu_0}{4\pi} \frac{\vec{V}}{\mathscr{r}} \int \rho(\vec{r}', t_r) d\tau'$$

Hence,

$$\boxed{\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\vec{V}}{(\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{V})} = \frac{\vec{V}}{c^2} V(\vec{r}, t)}$$

E1 Find potentials of a point charge moving with constant velocity

Suppose $\vec{W}(t) = t\vec{V}$

(Remark: $\frac{d\vec{W}}{dt} = \vec{V}$ so evaluation at retarded time is same as any other time, this notation does not interfere with L.W. potentials on ③)

Retarded time?

$$\| \vec{r} - t_r \vec{V} \| = c(t - t_r)$$

Square both sides,

$$(\vec{r} - t_r\vec{V}) \cdot (\vec{r} - t_r\vec{V}) = c^2(t^2 - 2tt_r + t_r^2)$$

$$r^2 - 2t_r \vec{V}\cdot\vec{r} + t_r^2 V^2 = c^2(t^2 - 2tt_r + t_r^2)$$

Consequently,

$$\underbrace{(c^2 - V^2)}_{A} t_r^2 + \underbrace{2(\vec{V}\cdot\vec{r} - c^2 t)}_{B} t_r + \underbrace{c^2t^2 - r^2}_{C} = 0$$

$$A t_r^2 + B t_r + C = 0$$

$$t_r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{2(c^2 t - \vec{V}\cdot\vec{r}) \pm \sqrt{4(\vec{V}\cdot\vec{r} - c^2 t)^2 - 4(c^2 - V^2)(c^2t^2 - r^2)}}{2(c^2 - V^2)}$$

$$= \frac{c^2 t - \vec{V}\cdot\vec{r} \pm \sqrt{(c^2 t - \vec{V}\cdot\vec{r})^2 + (c^2 - V^2)(r^2 - c^2 t^2)}}{c^2 - V^2}$$

GRIFFITH's suggests examining $V \longrightarrow 0$ to fix the sign, $t_r = t \pm \frac{r}{c}$ should have $t_r = t - r/c$ for point charge at rest at origin $\longleftrightarrow$ choose minus

E1 continued

We found the retarded time for $\vec{w}(t) = t\vec{v}$,

$$t_r = \frac{c^2 t - \vec{v}\cdot\vec{r} - \sqrt{(c^2 t - \vec{v}\cdot\vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}$$

Next, consider,

$$\mathscr{r} = c(t - t_r) \quad \text{and} \quad \hat{\mathscr{r}} = \frac{\vec{r} - \vec{v}t_r}{c(t - t_r)}$$

Consequently,

$$\begin{aligned}
\mathscr{r}\left(1 - \frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}\right) &= c(t - t_r)\left[1 - \frac{\vec{v}}{c}\cdot\left(\frac{\vec{r} - \vec{v}t_r}{c(t - t_r)}\right)\right] \\
&= c(t - t_r) - \frac{\vec{v}\cdot\vec{r}}{c} + \frac{\vec{v}\cdot\vec{v}t_r}{c} \\
&= \frac{1}{c}\left((c^2 t - \vec{r}\cdot\vec{v}) - (c^2 - v^2)t_r\right) \\
&= \frac{1}{c}\left((c^2 t - \vec{r}\cdot\vec{v}) - \left[\underbrace{c^2 t - \vec{v}\cdot\vec{r} - \sqrt{(c^2 t - \vec{v}\cdot\vec{r})^2 - (c^2 - v^2)(r^2 - c^2 t^2)}}_{\text{from quadratic formula on page 4}}\right]\right) \\
&= \frac{1}{c}\sqrt{(c^2 t - \vec{v}\cdot\vec{r})^2 - (c^2 - v^2)(r^2 - c^2 t^2)}
\end{aligned}$$

Therefore,

$$\boxed{\begin{aligned}
V(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0}\,\frac{qc}{\sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 - (c^2 - v^2)(r^2 - c^2 t^2)}} \\[2ex]
\vec{A}(r, t) &= \frac{\vec{v}}{c^2}V(\vec{r}, t) = \frac{q\vec{v}}{4\pi\epsilon_0 c\sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 - (c^2 - v^2)(r^2 - c^2 t^2)}}
\end{aligned}}$$

Remark: $\dfrac{q}{\epsilon_0 c} = \dfrac{qc}{\epsilon_0 c^2} = \dfrac{qc}{\epsilon_0\left(\frac{1}{\epsilon_0\mu_0}\right)} = \dfrac{qc\mu_0}{\cancel{\epsilon}}$