Lecture 34: $\vec{E}$ & $\vec{B}$ Fields of Point Charge

We found the Liénard-Wiechert potentials for a point charge q in arbitrary motion

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left(\frac{qc}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}\right) \quad \vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)}$$

We should review the terms and conditions here,

$\vec{w}(t)$ = position of q at time t

t_r is <u>retarded</u> time given by $\boxed{\|\vec{r}-\vec{w}(t_r)\| = c(t-t_r)}$

$$\underbrace{\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)}$$

separation vector from field point $\vec{r}$ (at time t) and retarded position of q

Our goal today (gulp) is to calculate $\vec{E}$ & $\vec{B}$ from the potentials by calculating

$$\boxed{\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t} \quad \vec{B} = \nabla\times\vec{A}}$$

It turns out,

$$\boxed{\begin{aligned}\vec{E}(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right] \\ \vec{B}(\vec{r},t) &= \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t)\end{aligned}}$$

- the magnetic field of a point charge is always $\perp$ to the $\vec{E}$, and to the vector from the retarded point.

$$V(\vec{r},t) = \underbrace{\frac{qc}{4\pi\epsilon_0}}_{k}\left(\frac{1}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}\right) \qquad (10.53)$$

Differentiate,

$$\nabla V = \frac{-k}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2}\nabla(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}) \qquad (10.56)$$

We have $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$ and $\vec{v} = \dot{\vec{w}}(t_r) = \frac{d\vec{w}}{dt_r}$

consequently $\mathscr{r} = c(t - t_r)$ and,

$$\nabla \mathscr{r} = -c\nabla t_r \qquad (10.57)$$

Next we work on $\nabla(\vec{\mathscr{r}}\cdot\vec{v})$ from 10.56, we need one of the less known product rules

$$(10.58) \quad \nabla(\vec{\mathscr{r}}\cdot\vec{v}) = \underbrace{(\vec{\mathscr{r}}\cdot\nabla)\vec{v}}_{1} + \underbrace{(\vec{v}\cdot\nabla)\vec{\mathscr{r}}}_{2} + \underbrace{\vec{\mathscr{r}}\times(\nabla\times\vec{v})}_{3} + \underbrace{\vec{v}\times(\nabla\times\vec{\mathscr{r}})}_{4}$$

Begin with 1,

$\vec{v}$ is function of t_r we usually omit.

$$(\vec{\mathscr{r}}\cdot\nabla)\vec{v} = \left(\mathscr{r}_x\frac{\partial}{\partial x} + \mathscr{r}_y\frac{\partial}{\partial y} + \mathscr{r}_z\frac{\partial}{\partial z}\right)\vec{v}(t_r)$$

$$= \mathscr{r}_x\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial z}$$

$$= \frac{d\vec{v}}{dt_r}\left(\mathscr{r}_x\frac{\partial t_r}{\partial x} + \mathscr{r}_y\frac{\partial t_r}{\partial y} + \mathscr{r}_z\frac{\partial t_r}{\partial z}\right)$$

$$= \underline{\vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)} \quad 1$$

$$\boxed{\text{Def}^{\underline{n}}/ \quad \vec{a} = \frac{d\vec{v}}{dt_r} \quad \text{where} \quad \vec{v} = \frac{d\vec{w}}{dt_r}}$$

acceleration also at retarded time.

Recall $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$ hence ② can be analyzed,

$$(\vec{v}\cdot\nabla)\vec{\mathscr{r}} = (\vec{v}\cdot\nabla)(\vec{r} - \vec{w}(t_r))$$

$$\begin{aligned}
(\vec{v}\cdot\nabla)\vec{r} &= (\vec{v}\cdot\nabla)\vec{r} \\
&= (\vec{v}\cdot\nabla)(x\hat{x} + y\hat{y} + z\hat{z}) \\
&= v_x\partial_x\vec{r} + v_y\partial_y\vec{r} + v_z\partial_z\vec{r} \\
&= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \\
&= \vec{v}
\end{aligned}$$

$$\begin{aligned}
(\vec{v}\cdot\nabla)\vec{w} &= (v_x\partial_x + v_y\partial_y + v_z\partial_z)(\vec{w}(t_r)) \\
&= v_x\frac{d\vec{w}}{dt_r}\frac{\partial t_r}{\partial x} + v_y\frac{d\vec{w}}{dt_r}\frac{\partial t_r}{\partial y} + v_z\frac{d\vec{w}}{dt_r}\frac{\partial t_r}{\partial z} \\
&= \left(\frac{d\vec{w}}{dt_r}\right)\left[v_x\frac{\partial t_r}{\partial x} + v_y\frac{\partial t_r}{\partial y} + v_z\frac{\partial t_r}{\partial z}\right] \\
&= \vec{v}(\vec{v}\cdot\nabla t_r)
\end{aligned}$$

In summary for ② we find,

$$(\vec{v}\cdot\nabla)\vec{\mathscr{r}} = \vec{v} - (\vec{v}\cdot\nabla t_r)\vec{v}$$

Continuing, let's attack $\vec{\mathscr{r}} \times (\nabla \times \vec{v})$ ③ by examining,

$$\nabla \times \vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{x} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{y} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z}$$

$$= \left(\frac{dV_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dV_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{x} + \left(\frac{dV_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dV_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{y}$$

$$+ \left(\frac{dV_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dV_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{z}$$

$$= -\left\langle \frac{dV_x}{dt_r}, \frac{dV_y}{dt_r}, \frac{dV_z}{dt_r} \right\rangle \times \left\langle \frac{\partial t_r}{\partial x}, \frac{\partial t_r}{\partial y}, \frac{\partial t_r}{\partial z} \right\rangle$$

$$= -\vec{a} \times \nabla t_r$$

Finally move onto ④ and analyze $\nabla \times \vec{\mathscr{r}}$ where $\vec{\mathscr{r}} = \vec{r} - \vec{w}$ and since $\nabla \times \vec{r} = 0$ we find,

$$\nabla \times \vec{\mathscr{r}} = \nabla \times (\vec{r} - \vec{w})$$

$$= -\nabla \times \vec{w}$$

$$= -\left[\left(\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z}\right)\hat{x} + \left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right)\hat{y} + \left(\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y}\right)\hat{z}\right]$$

$$= -\left[-\vec{v} \times \nabla t_r\right]$$

$$= \vec{v} \times \nabla t_r$$

(by same calculation as we just completed for $\nabla \times \vec{V}$ just replace $\vec{v}$ with $\vec{w}$)

Let's collect our calculations from 1, 2, 3, 4

$$\nabla(\vec{\mathscr{r}}\cdot\vec{v}) = (\vec{\mathscr{r}}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{\mathscr{r}} + \vec{\mathscr{r}}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{\mathscr{r}})$$

$$= \underbrace{\vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)}_{1} + \underbrace{\vec{v} - \vec{v}(\vec{v}\cdot\nabla t_r)}_{2} \underbrace{- \vec{\mathscr{r}}\times(\vec{a}\times\nabla t_r)}_{3} + \underbrace{\vec{v}\times(\vec{v}\times\nabla t_r)}_{4}$$

$$= \cancel{\vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)}^{1} + \vec{v} - \cancel{\vec{v}(\vec{v}\cdot\nabla t_r)}^{2}$$
$$- \cancel{\vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)}^{1} + \nabla t_r(\vec{\mathscr{r}}\cdot\vec{a}) + \cancel{\vec{v}(\vec{v}\cdot\nabla t_r)}^{2} - (\nabla t_r)(\vec{v}\cdot\vec{v})$$

$$= \vec{v} + (\vec{\mathscr{r}}\cdot\vec{a} - v^2)\nabla t_r \qquad (10.65)$$

Therefore, recalling $\nabla\mathscr{r} = -c\nabla t_r$ we calculate,

$$\nabla V = \frac{qc}{4\pi\epsilon_0}\,\frac{-1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2}\,\nabla(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})$$

$$= \frac{qc}{4\pi\epsilon_0}\,\frac{-1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2}\left[-c^2\nabla t_r - \vec{v} - (\vec{\mathscr{r}}\cdot\vec{a} - v^2)\nabla t_r\right]$$

$$= \frac{qc}{4\pi\epsilon_0}\,\frac{1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2}\left[\vec{v} + (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\boxed{\nabla t_r}\right]$$

Lemma

$$= \frac{qc}{4\pi\epsilon_0}\,\frac{1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2}\left[\vec{v} + (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\boxed{\frac{-\vec{\mathscr{r}}}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}}\right]$$

(10.69) $\rightarrow$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^3}\left[(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})\vec{v} - (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\vec{\mathscr{r}}\right]$$

$$\boxed{\underline{\text{Lemma}}:\ \nabla t_r = \frac{-\vec{\mathscr{r}}}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}}$$

$$\boxed{\textit{Lemma:}\quad \nabla t_r = \frac{-\vec{\mathscr{r}}}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}}$$

Proof:

$$-c\nabla t_r = \nabla \mathscr{r} = \nabla\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = \frac{1}{2\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}}}\nabla(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}})$$

$$-c\nabla t_r = \frac{1}{2\mathscr{r}}\left(\vec{\mathscr{r}}\times(\nabla\times\vec{\mathscr{r}}) + \vec{\mathscr{r}}\times(\nabla\times\vec{\mathscr{r}}) + (\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}} + (\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}}\right)$$

$$= \frac{1}{\mathscr{r}}\left(\vec{\mathscr{r}}\times(\underbrace{\nabla\times\vec{\mathscr{r}}}_{\text{pg. 4}}) + \underbrace{(\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}}}_{\text{sublemma}}\right)$$

$$= \frac{1}{\mathscr{r}}\left(\vec{\mathscr{r}}\times(\vec{v}\times\nabla t_r) + \vec{\mathscr{r}} - \vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r)\right)$$

Sublemma: $(\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}} = \vec{\mathscr{r}} - \vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r)$

$$(\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}} = (\mathscr{r}_x\partial_x + \mathscr{r}_y\partial_y + \mathscr{r}_z\partial_z)(\vec{r} - \vec{w})$$

$$= \mathscr{r}_x\hat{x} + \mathscr{r}_y\hat{y} + \mathscr{r}_z\hat{z} - \mathscr{r}_x\partial_x\vec{w} - \mathscr{r}_y\partial_y\vec{w} - \mathscr{r}_z\partial_z\vec{w}$$

$$= \vec{\mathscr{r}} - \mathscr{r}_x\frac{d\vec{w}}{dt_r}\frac{\partial t_r}{\partial x} - \mathscr{r}_y\frac{d\vec{w}}{dt_r}\frac{\partial t_r}{\partial y} - \mathscr{r}_z\frac{d\vec{w}}{dt_r}\frac{\partial t_r}{\partial z}$$

$$= \vec{\mathscr{r}} - (\vec{\mathscr{r}}\cdot\nabla t_r)\vec{v}. \;//$$

Hence,

$$-c\nabla t_r = \frac{1}{\mathscr{r}}\left(\cancel{\vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r)} - \nabla t_r(\vec{\mathscr{r}}\cdot\vec{v}) + \vec{\mathscr{r}} - \cancel{\vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r)}\right)$$

$$= \frac{1}{\mathscr{r}}\left(\vec{\mathscr{r}} - (\vec{\mathscr{r}}\cdot\vec{v})\nabla t_r\right)$$

$$\Rightarrow (\vec{\mathscr{r}}\cdot\vec{v} - c\mathscr{r})\nabla t_r = \vec{\mathscr{r}}$$

$$\therefore\quad \nabla t_r = \frac{-\vec{\mathscr{r}}}{c\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v}} \;//$$

Problem 10.20 yields

$$\frac{\partial \vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_0}\,\frac{qc}{(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})^3}\Big[(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})(-\vec{v}+\mathscr{r}\vec{a}/c) + \frac{\mathscr{r}}{c}(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{v}\Big]$$

Therefore, as $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$

we calculate,

$$\vec{E} = \frac{-1}{4\pi\epsilon_0}\,\frac{qc}{(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})^3}\Big[\overbrace{\cancel{(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})\vec{v}} - (c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{\mathscr{r}}}^{10.69 \text{ from } 5} + (\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})(\cancel{-\vec{v}}+\mathscr{r}\vec{a}/c) + \frac{\mathscr{r}}{c}(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{v}\Big]$$

Thus,

$$\vec{E} = \frac{qc}{4\pi\epsilon_0(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})^3}\Big[(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{\mathscr{r}} - \frac{\mathscr{r}}{c}(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{v} - (\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})\mathscr{r}\vec{a}/c\Big]$$

$$\vec{E} = \frac{qc}{4\pi\epsilon_0(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})^3}\Big[(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\left(\vec{\mathscr{r}} - \mathscr{r}\vec{v}/c\right) - \underbrace{(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})\mathscr{r}\vec{a}/c}_{*}\Big]$$

Let $\boxed{\vec{u} = c\hat{\mathscr{r}} - \vec{v}}$ ← definition of $\vec{u}$

Notice $\vec{\mathscr{r}} - \mathscr{r}\vec{v}/c = \mathscr{r}\hat{\mathscr{r}} - \mathscr{r}\vec{v}/c = \frac{\mathscr{r}}{c}(c\hat{\mathscr{r}} - \vec{v}) = \frac{\mathscr{r}}{c}\vec{u}$

Therefore,

$$\vec{E} = \frac{qc}{4\pi\epsilon_0(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})^3}\Big[(c^2-v^2)\frac{\mathscr{r}\vec{u}}{c} + (\vec{\mathscr{r}}\cdot\vec{a})\frac{\vec{u}\mathscr{r}}{c} - \underbrace{(\mathscr{r}^2 - \mathscr{r}\vec{\mathscr{r}}\cdot\vec{v}/c)\vec{a}}_{*}\Big]$$

$$= \frac{qc}{4\pi\epsilon_0(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})^3}\Big[(c^2-v^2)\frac{\vec{u}\mathscr{r}}{c} + \frac{\vec{u}\mathscr{r}}{c}(\vec{\mathscr{r}}\cdot\vec{a}) - \frac{\mathscr{r}}{c}(\vec{\mathscr{r}}\cdot\vec{u})\vec{a}\Big]$$

$$\boxed{\begin{aligned} \mathscr{r}^2 - \mathscr{r}\vec{\mathscr{r}}\cdot\vec{v}/c &= \vec{\mathscr{r}}\cdot(\vec{\mathscr{r}} - \mathscr{r}\vec{v}/c) \\ &= \frac{1}{c}\vec{\mathscr{r}}\cdot(c\mathscr{r}\hat{\mathscr{r}} - \vec{v}\mathscr{r}) \\ &= \frac{\mathscr{r}}{c}\vec{\mathscr{r}}\cdot(c\hat{\mathscr{r}} - \vec{v}) \end{aligned}}$$

Continuing, notice the common $\mathscr{r}/c$ factor whose rearrangement yields c cancelling to give

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^3}\left[(c^2 - v^2)\vec{u} + \vec{u}(\vec{\mathscr{r}}\cdot\vec{a}) - \vec{a}(\vec{\mathscr{r}}\cdot\vec{u})\right]$$

Notice $\boxed{\vec{u} = c\hat{\mathscr{r}} - \vec{v}} \Rightarrow \vec{\mathscr{r}}\cdot\vec{u} = c\hat{\mathscr{r}}\cdot\vec{\mathscr{r}} - \vec{\mathscr{r}}\cdot\vec{v} = c\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v}$ and applying the "BAC- CAB" identity we derive,

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\underbrace{\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}}\Big[\underbrace{(c^2 - v^2)\vec{u}}_{\substack{\text{gives the}\\ \text{generalized}\\ \text{Coulomb Field}\\ \propto \frac{1}{\mathscr{r}^2}}} + \underbrace{\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})}_{\substack{\text{gives the}\\ \text{radiation field}\\ \propto \frac{1}{\mathscr{r}}}}\Big]}$$

The magnetic field is found from the curl of the vector potential, recall $\vec{A}(\vec{r},t) = \frac{1}{c^2}\vec{v}\,V(\vec{r},t)$ hence,

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{c^2}\nabla\times(V\vec{v})$$

a product rule for curl,

$$= \frac{1}{c^2}\left[V(\nabla\times\vec{v}) - \vec{v}\times(\nabla V)\right]$$

$$= \frac{1}{c^2}\left[\frac{qc}{4\pi\epsilon_0\,\vec{\mathscr{r}}\cdot\vec{u}}\left(\nabla t_r\times\vec{a}\right) - \vec{v}\times\left(\frac{qc}{4\pi\epsilon_0(\vec{\mathscr{r}}\cdot\vec{u})^2}\left[\vec{v} + (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\nabla t_r\right]\right)\right]$$

$$= \frac{1}{c}\frac{q}{4\pi\epsilon_0}\left[\left(\frac{-\vec{\mathscr{r}}}{(\vec{\mathscr{r}}\cdot\vec{u})^2}\right)\times\vec{a} - \frac{1}{(\vec{\mathscr{r}}\cdot\vec{u})^2}(c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\,\vec{v}\times\left(\frac{-\vec{\mathscr{r}}}{\vec{\mathscr{r}}\cdot\vec{u}}\right)\right]$$

$$= \frac{-1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\vec{\mathscr{r}}\times\left[(\vec{\mathscr{r}}\cdot\vec{u})\vec{a} + (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\vec{v}\right]$$

$$= \frac{-1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\vec{\mathscr{r}}\times\Big[\underbrace{(c^2 - v^2)\vec{v} + (\vec{\mathscr{r}}\cdot\vec{a})\vec{v}}_{\text{can replace } \vec{v} \text{ with } -\vec{u} \text{ since}} + (\vec{\mathscr{r}}\cdot\vec{u})\vec{a}\Big]$$

$$\vec{\mathscr{r}}\times\vec{u} = \vec{\mathscr{r}}\times(c\hat{\mathscr{r}} - \vec{v}) = -\vec{\mathscr{r}}\times\vec{v}$$

Continuing to simplify $\nabla \times \vec{A}$, $\vec{u}\,(\vec{\boldsymbol{\mathscr{r}}}\cdot\vec{a}) - \vec{a}\,(\vec{\boldsymbol{\mathscr{r}}}\cdot\vec{u}) = \vec{\boldsymbol{\mathscr{r}}}\times(\vec{u}\times\vec{a})$

$$\vec{B} = \frac{-1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\vec{\boldsymbol{\mathscr{r}}}\cdot\vec{u})^3}\vec{\boldsymbol{\mathscr{r}}}\times\left[-(c^2-v^2)\vec{u} + \vec{a}(\vec{\boldsymbol{\mathscr{r}}}\cdot\vec{u}) - \vec{u}(\vec{\boldsymbol{\mathscr{r}}}\cdot\vec{a})\right]$$

$$= \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\left[\frac{q\,\boldsymbol{\mathscr{r}}}{(\vec{\boldsymbol{\mathscr{r}}}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\boldsymbol{\mathscr{r}}}\times(\vec{u}\times\vec{a})\right]\right]\frac{1}{4\pi\epsilon_0}$$

$$= \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\vec{E}$$

That is,

$$\boxed{\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\vec{E}(\vec{r},t)} \qquad (10.73)$$

Finally as an application of our formulae, let us write the force on Q due to the q undergoing the motion described thus far by $w(t)$,

$$\vec{F} = Q(\vec{E} + \vec{v}_Q\times\vec{B})$$

$$\vec{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\boldsymbol{\mathscr{r}}}{(\vec{\boldsymbol{\mathscr{r}}}\cdot\vec{u})^3}\Big\{(c^2-v^2)\vec{u} + \vec{\boldsymbol{\mathscr{r}}}\times(\vec{u}\times\vec{a})$$

$$+ \frac{\vec{v}_Q}{c}\times\left[\hat{\boldsymbol{\mathscr{r}}}\times\left((c^2-v^2)\vec{u} + \vec{\boldsymbol{\mathscr{r}}}\times(\vec{u}\times\vec{a})\right)\right]\Big\}$$

GRIFFITHS SUGGESTS THIS FORMULA PAIRED WITH SUPERPOSITION GIVES THE ENTIRE THEORY OF CLASSICAL ELECTRODYNAMICS.

(keep in mind $\vec{a}, \vec{\boldsymbol{\mathscr{r}}}, \vec{u}, \vec{v}$ are all evaluated at t_r)