LECTURE 37: SPECIAL RELATIVITY & E&M.

We begin with the story of Lorentz Transformations, actually, before that **SPACETIME**

Def^n/ $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ are spacetime coordinates measured with respect to a fixed inertial frame of reference; x^μ, $\mu = 0,1,2,3$

Two inertial frames of reference for spacetime must be related by a Lorentz Transformation. Most commonly we consider the x-boost

$$\bar{x}^0 = \gamma(x^0 - \beta x^1)$$
$$\bar{x}^1 = \gamma(x^1 - \beta x^0)$$
$$\bar{x}^2 = x^2$$
$$\bar{x}^3 = x^3$$

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \underbrace{\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Lorentz Matrix}} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Notice $c\bar{t} = \gamma(ct - \frac{v}{c}x) \Rightarrow \bar{t} = \gamma(t - vx/c^2)$ whereas $\bar{x} = \gamma(-\frac{v}{c}tc + x) = \gamma(x - vt)$ may be familiar from the modern physics course (?).

Def^n/ $\overline{V} = (V^0, \vec{V}) = (V^0, V^1, V^2, V^3)$ and the Minkowski product of two such 4-vectors $\overline{V}, \overline{W}$ is given by

$$\overline{V} \cdot \overline{W} = V^0 W^0 - \vec{V} \cdot \vec{W}$$
$$= V^0 W^0 - V^1 W^1 - V^2 W^2 - V^3 W^3$$

The Minkowski product gives rise to several constructions which will be crucial to formulate physical law within the context of Special Relativity. Einstein's Axioms

(1.) speed of light is constant in all inertial frames of reference
(2.) laws of physics have same form in all frames of reference (inertially related)

Axiom (1.) is enforced by Lorentz Transformations and we'll see (2.) is greatly assisted by the use of contravariant and covariant tensors.

Def$^{\underline{n}}$/ $g_{\mu\nu} = -\delta_{\mu,0}\delta_{\nu,0} + \delta_{\mu,1}\delta_{\nu,1} + \delta_{\mu,2}\delta_{\nu,2} + \delta_{\mu,3}\delta_{\nu,3}$ defines the <u>Minkowski</u> <u>Metric</u>. As matrix, $[g_{\mu\nu}] = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ and note

$$\begin{aligned} \overline{A}\cdot\overline{B} &= -A^0B^0 + \vec{A}\cdot\vec{B} \\ &= [A^0, A^1, A^2, A^3]\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{bmatrix} \\ &= \sum_{\mu,\nu=0}^{3} A^\mu g_{\mu\nu} B^\nu \\ &= \overline{A}^T g \overline{B} \quad \text{(row - matrix - column product)} \end{aligned}$$

The above is simply notation, and we should finish this with a labor-saving customary notation

Def$^{\underline{n}}$/ Einstein summation notation: up/down repeated indices are summed over. For example,

$$\overline{A}\cdot\overline{B} = A^\mu g_{\mu\nu} B^\nu = g_{\mu\nu} A^\mu B^\nu$$

Index Raising & Lowering in Minkowski Space

Minkowski Space is $\mathbb{R}^4$ paired with the Minkowski Metric we define Lorentz Transformations to be the linear transformations which preserve the Minkowski product of 4-vectors. A Lorentz matrix is the matrix of a Lorentz transformation and we speak of these interchangeably.

Def^{n}/ $T: \mathbb{R}^4 \to \mathbb{R}^4$ linear map with $T(\bar{A}) \cdot T(\bar{B}) = \bar{A} \cdot \bar{B}$ for all $\bar{A}, \bar{B} \in \mathbb{R}^4$ is Lorentz Transformation. We say $SO(1,3) = \{ \Lambda \mid \Lambda^T g \Lambda = g \}$ is the set of Lorentz Matrices

A short calculation is needed to understand the above def^{n};
Suppose $T(\bar{x}) = \Lambda \bar{x}$ for a Lorentz Transformation T,

$$
\begin{aligned}
T(\bar{x}) \cdot T(\bar{y}) = \bar{x} \cdot \bar{y} &\iff (\Lambda \bar{x}) \cdot (\Lambda \bar{y}) = \bar{x} \cdot \bar{y} \\
&\iff (\Lambda \bar{x})^T g \Lambda \bar{y} = \bar{x}^T g \bar{y} \\
&\iff \bar{x}^T \Lambda^T g \Lambda \bar{y} = \bar{x}^T g \bar{y} \\
&\iff \underline{\Lambda^T g \Lambda = g}.
\end{aligned}
$$

Def^{n}/ Two coordinate systems x^μ and $\bar{x}^{\mu'}$ are <u>inertially related</u> if there exists $\Lambda \in SO(1,3)$ for which $\bar{x} = \Lambda x$ which in terms of index calculus,

$$\bar{x}^{\mu'} = \Lambda^{\mu'}_{\mu} x^\mu$$

Likewise, $x^\mu = \Lambda^{\mu}_{\mu'} \bar{x}^{\mu'}$.

Notice $\Lambda^{\mu'}_{\mu}$ and $\Lambda^{\mu}_{\mu'}$ are different matrices (usually) and,

$$\underline{\bar{x}^{\mu'}} = \Lambda^{\mu'}_{\mu} x^\mu = \Lambda^{\mu'}_{\mu} \Lambda^{\mu}_{\nu'} \bar{x}^{\nu'} = \underline{\delta^{\mu'}_{\nu'} \bar{x}^{\nu'}}$$

see what I did here?

Consequently, as the above holds for all $\bar{x}^{\nu'}$, $\boxed{\Lambda^{\mu'}_{\mu} \Lambda^{\mu}_{\nu'} = \delta^{\mu'}_{\nu'}}$

If $\Lambda = (\Lambda^{\mu'}_{\ \mu})$ then $\Lambda^{-1} = (\Lambda^{\mu}_{\ \mu'})$ since we just calculated that

$$\Lambda^{\mu'}_{\ \mu} \Lambda^{\mu}_{\ \nu'} = \delta^{\mu'}_{\nu'} = \begin{cases} 1 & \text{if } \mu' = \nu' \\ 0 & \text{if } \mu' \neq \nu' \end{cases}$$

I'll use the custom that the index raising/lowering does not apply to Kronecker Delta's, but I'll raise/lower indices to conserve index height in equations like the above.

Def^{n}/ A <u>contravariant</u> 4-vector is an object which transforms like coordinates of Minkowski space. That is A^{μ} defines <u>contravariant</u> 4-vector if $\bar{x}^{\mu'} = \Lambda^{\mu'}_{\ \nu} x^{\nu}$ then $\bar{A}^{\mu'} = \Lambda^{\mu'}_{\ \nu} A^{\nu}$. Likewise, a <u>covariant</u> 4-vector is an object which transforms inversely to coordinates; B_{μ} is covariant 4-vector if when $\bar{x}^{\mu'} = \Lambda^{\mu'}_{\ \nu} x^{\nu}$ we have $\bar{B}_{\mu'} = \Lambda^{\nu}_{\ \mu'} B_{\nu}$.

There will be examples of these in the next Lecture like 4-momentum or the proper-4-velocity etc. I'm focuse on the math set-up here.

Def^{n}/ Given contravariant 4-vector A^{μ} we define the <u>covariant</u> components of A^{μ} via

$$A_{\mu} = g_{\mu\nu} A^{\nu}$$

Likewise, given a covariant 4-vector B_{μ} we define the <u>contravariant</u> components of B_{μ} via

$$B^{\mu} = g^{\mu\nu} B_{\nu}$$

where $g^{\mu\nu} g_{\nu\alpha} = \delta^{\mu}_{\alpha}$, which just means $(g^{\mu\nu}) = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

(it's more interesting in G.R. where g is coordinate dependent)

Thm/ Suppose A^μ and B^μ are contravariant 4-vectors and suppose x^μ and $\bar{x}^{\mu'}$ are inertially related then

(1.) $\bar{A} \cdot \bar{B} = A_\mu B^\mu = A^\mu B_\mu$

(2.) $\bar{A}^{\mu'} \bar{B}_{\mu'} = A^\mu B_\mu$

<u>Proof</u>: $\bar{x} = \Lambda x$ for $\Lambda \in SO(1,3)$ where $\Lambda^T g \Lambda = g$

where $g = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ thus $gg = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ hence

$$(g \Lambda^T g)\Lambda = g^2 = I \Rightarrow \underline{\Lambda^{-1} = g \Lambda^T g}.$$

In index notation, we set-up $\bar{x}^{\mu'} = \Lambda^{\mu'}_{\nu} x^\nu$ so expressing transpose is ackward. I'll go back to basics to circumvent this trouble,

$$\bar{x} \cdot \bar{y} = \underbrace{g_{\mu\nu} x^\mu y^\nu = g_{\mu'\nu'} \bar{x}^{\mu'} \bar{y}^{\nu'}} = \bar{\bar{x}} \cdot \bar{\bar{y}}$$

$$\begin{aligned} g_{\mu\nu} x^\mu y^\nu &= g_{\mu'\nu'} (\Lambda^{\mu'}_{\mu} x^\mu)(\Lambda^{\nu'}_{\nu} y^\nu) \\ &= g_{\mu'\nu'} \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} x^\mu y^\nu \end{aligned}$$

$$\therefore \quad \boxed{g_{\mu\nu} = \Lambda^{\mu'}_{\mu} g_{\mu'\nu'} \Lambda^{\nu'}_{\nu}}$$

Proof of (1) is simple, $\bar{A} \cdot \bar{B} = g_{\mu\nu} A^\mu B^\nu = A_\nu B^\nu$ and likewise $\bar{A} \cdot \bar{B} = g_{\mu\nu} A^\mu B^\nu = A^\mu g_{\mu\nu} B^\nu = A^\mu B_\mu$. $(g_{\mu\nu} = g_{\nu\mu})$

Next, to prove (2.),

$$\begin{aligned} A_\mu B^\mu = g_{\mu\nu} A^\nu B^\mu &= \Lambda^{\mu'}_{\mu} g_{\mu'\nu'} \Lambda^{\nu'}_{\nu} A^\nu B^\mu \\ &= g_{\mu'\nu'} (\Lambda^{\nu'}_{\nu} A^\nu)(\Lambda^{\mu'}_{\mu} B^\mu) \\ &= g_{\mu'\nu'} \bar{A}^{\nu'} \bar{B}^{\mu'} \\ &= \bar{A}_{\mu'} \bar{B}^{\mu'}. // \end{aligned}$$

Another way to see the invariance of the contraction of a contravariant & covariant index is given by calculus and the chain-rule,

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^{\nu} \quad \& \quad x^{\mu} = \Lambda^{\mu}_{\nu'} \bar{x}^{\nu'}$$

$$\frac{\partial x^{\mu'}}{\partial x^{\alpha}} = \Lambda^{\mu'}_{\nu} \frac{\partial x^{\nu}}{\partial x^{\alpha}} = \Lambda^{\mu'}_{\nu} \delta^{\nu}_{\alpha} = \Lambda^{\mu'}_{\alpha}$$

$$\frac{\partial x^{\mu}}{\partial \bar{x}^{\alpha'}} = \frac{\partial}{\partial \bar{x}^{\alpha'}}\left(\Lambda^{\mu}_{\nu'} \bar{x}^{\nu'}\right) = \Lambda^{\mu}_{\nu'} \frac{\partial \bar{x}^{\nu'}}{\partial \bar{x}^{\alpha'}} = \Lambda^{\mu}_{\nu'} \delta^{\nu'}_{\alpha'} = \Lambda^{\mu}_{\alpha'}$$

$$\underbrace{\frac{\partial x^{\mu}}{\partial \bar{x}^{\alpha'}} \frac{\partial \bar{x}^{\alpha'}}{\partial x^{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\nu}}}_{\text{chain-rule}} = \delta^{\mu}_{\nu}$$

$$\Lambda^{\mu}_{\alpha'} \Lambda^{\alpha'}_{\nu} = \delta^{\mu}_{\nu}$$

Ok, so now to the contraction. Suppose $A^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} A^{\nu}$ and $B_{\mu'} = \frac{\partial x^{\nu}}{\partial x^{\mu'}} B_{\nu}$ then

$$\begin{aligned} A^{\mu'} B_{\mu'} &= \left(\frac{\partial x^{\mu'}}{\partial x^{\nu}} A^{\nu}\right)\left(\frac{\partial x^{\sigma}}{\partial x^{\mu'}} B_{\sigma}\right) \\ &= \left(\frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\sigma}}{\partial x^{\mu'}}\right) A^{\nu} B_{\sigma} \\ &= \delta^{\sigma}_{\nu} A^{\nu} B_{\sigma} \\ &= A^{\sigma} B_{\sigma} \end{aligned}$$

← should only use ν once so I introduced σ to avoid confusion of the distinct sums.

Remark: calculation on this page is more general whereas last page used $\Lambda^T g \Lambda = g$ which is special.

Invariant Interval & Spacetime Translations

A point (ct, x, y, z) in Minkowski Space is an <u>event</u>. Given two events x^μ and y^μ we can define the <u>invariant</u> <u>interval</u> between the events,

$$\overline{(y-x)} \cdot \overline{(y-x)} = (y^\mu - x^\mu)(y_\mu - x_\mu)$$
$$= -(y^0 - x^0)^2 + (y^1 - x^1)^2 + (y^2 - x^2)^2 + (y^3 - x^3)^2$$

In other words, if $(x_2)^\mu$ and $(x_1)^\mu$ are events then

$$((x_2 - x_1)^\mu) = (c(t_2 - t_1), x_2 - x_1, y_2 - y_1, z_2 - z_1)$$
$$= (c\Delta t, \Delta x, \Delta y, \Delta z)$$

and the invariant interval between the events

$$\boxed{I = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

The invariant interval is the same for inertially related frames of reference. Consequently,

$$\boxed{\text{Th}^{\underline{m}}/ \ -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = -c^2(\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2}$$

<u>Proof</u>: $(y-x)^\mu (y-x)_\mu = (\bar{y} - \bar{x})^{\mu'} (\bar{y} - \bar{x})_{\mu'}$.

The above Th$^{\underline{m}}$ destroys the Newtonian concept of simultaneity and universal time shared amongst inertially related observers. However, the interval is invariant to <u>more</u> than just Lorentz Transformations.

Def$^{\underline{n}}$/ $T_a : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T_a(x) = x + a$ defines a spacetime translation; $\bar{x}^{\mu'} = x^\mu + a^\mu$

Th$^{\underline{m}}$/ The invariant interval is fixed under spacetime translations

<u>Proof</u>: Consider two events y^μ and x^μ then translate $y^{\mu'} = y^\mu + a^\mu$ and $x^{\mu'} = x^\mu + a^\mu \Rightarrow y^{\mu'} - x^{\mu'} = (y^\mu + a^\mu) - (x^\mu + a^\mu) = y^\mu - x^\mu$.

Poincaré Group

The set of origin-fixing Lorentz Transformations paired with the set of spacetime translations forms the <u>POINCARE GROUP</u>. These are all the transformations which leave the invariant interval between all pairs of events in Minkowski Space fixed.

$$\boxed{\begin{array}{l} \text{Def}^{\underline{n}}\ \ \mathcal{I}(x,y) = \overline{(y-x)} \cdot \overline{(y-x)} = -c^2\Delta t^2 + \|\Delta\vec{r}\|^2 \\ \quad \text{where } y - x = (c\Delta t, \Delta\vec{r}). \text{ Then,} \\ \qquad \mathcal{P} = \{ T : \mathbb{R}^4 \to \mathbb{R}^4 \mid \mathcal{I}(T(x), T(y)) = \mathcal{I}(x,y)\ \forall x,y \} \end{array}}$$

Naturally we can view $SO(1,3) \subset \mathcal{P}$ if we identify Lorentz Transformations with their matrices.

> **<u>CLAIM</u>:** rotations of space fixing time are Lorentz Transformations, also, x-boosts, y-boosts and z-boosts are Lorentz transformations (and hence in $\mathcal{P}$)

<u>Proof</u>: probably homework, a.k.a. TEST 3.//

$$\underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{x\text{-boost}} \quad \underbrace{\begin{pmatrix} \gamma\beta & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{y\text{-boost}} \quad \underbrace{\begin{pmatrix} \gamma\beta & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma\beta \end{pmatrix}}_{z\text{-boost}}$$

$$\underbrace{\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right)}_{\text{spatial rotation}} \quad \text{where } R^T R = I$$

(All solve $\Lambda^T g \Lambda = g$)
$\Lambda \in SO(1,3)$.