Lecture 38 : 4-Vectors and Physics

In this talk I'd like to introduce the basic variables needed to study motion of objects in Special Relativity. We continue using the 4 - vector formalism and we now introduce the infinitesimal interval

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2}$$

Consider the trajectory of some object and imagine a comoving coordinate system then $dx = dy = dz = 0$

Def^{n} $\longrightarrow$ $\boxed{\begin{array}{ll} X = X(\tau) & : \underline{\text{world}}\ \underline{\text{line}} \text{ parametrized} \\ ds^2 = -c^2 d\tau^2 & \quad \text{using the proper time } \tau \text{ for the object} \end{array}}$

Then calculate,

$$d\tau^2 = \frac{ds^2}{-c^2} = \frac{-c^2dt^2 + dx^2 + dy^2 + dz^2}{-c^2}$$

$$\frac{d\tau^2}{dt^2} = 1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = 1 - u^2/c^2$$

$$\boxed{\text{Def}^{n}/\ \vec{u} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \text{ and } \gamma(u) = \frac{1}{\sqrt{1 - u^2/c^2}}}$$

Observe $\dfrac{d\tau}{dt} = \sqrt{1 - u^2/c^2} \Rightarrow \dfrac{d\tau}{\sqrt{1-u^2/c^2}} = dt$

$$\therefore \boxed{dt = \gamma(u)\, d\tau}$$

$$\boxed{\text{Def}^{n}/\ U^{\mu} = \frac{dX^{\mu}}{d\tau} \text{ is the FOUR-VELOCITY OF THE OBJECT DISCUSSED ABOVE}}$$

We found $\frac{dt}{d\tau} = \gamma(u) = \frac{1}{\sqrt{1-u^2/c^2}}$ thus calculate,

$$U^\mu = \frac{dX^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dX^\mu}{dt} = \gamma(u)\frac{dX^\mu}{dt}$$

$$(U^\mu) = \gamma(u)\frac{d}{dt}\left(ct, x, y, z\right)$$

$$(U^\mu) = \left(c\gamma(u), \gamma(u)\frac{dx}{dt}, \gamma(u)\frac{dy}{dt}, \gamma(u)\frac{dz}{dt}\right)$$

$$\boxed{\overline{U} = (\gamma c, \gamma\vec{u}) \text{ where } \gamma = \frac{1}{\sqrt{1-u^2/c^2}}}$$

<u>Remark</u>: the quantity $\gamma\vec{u}$ is the <u>relativistic</u> <u>velocity</u>. It's simply the ordinary velocity $\vec{u}$ multiplied by $\gamma(u)$.

The Four Velocity $U^\mu = \frac{dX^\mu}{d\tau}$ is a 4-vector thus $U^\mu U_\mu$ is an <u>invariant</u>

$$\begin{aligned}
\overline{U}\cdot\overline{U} &= -\gamma^2c^2 + \gamma^2\vec{u}\cdot\vec{u} \\
&= \gamma^2(u^2-c^2) \\
&= \frac{u^2-c^2}{1-u^2/c^2} \\
&= -c^2\left(\frac{1-u^2/c^2}{1-u^2/c^2}\right) \\
&= -c^2
\end{aligned}$$

same value for any choice of approved coordinates. Easy enough to believe here, surprisingly, this type of observation does solve real world nontrivial problems...

4- ACCELERATION

We continue our study of an object with proper time τ

$$\boxed{\text{Def}^{n}/ \quad A^{\mu} = \frac{dU^{\mu}}{d\tau} = \frac{d^2 X^{\mu}}{d\tau^2}}$$

Let us denote the velocity by $\vec{u} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle = \frac{d\vec{r}}{dt}$ and ordinary acceleration by $\vec{a} = \frac{d\vec{u}}{dt} = \frac{d^2\vec{r}}{dt^2}$ then since $\frac{dt}{d\tau} = \gamma(u) = \frac{1}{\sqrt{1-u^2/c^2}}$ we find,

$$\bar{A} = \frac{d\bar{U}}{d\tau} = \frac{dt}{d\tau}\frac{d\bar{U}}{dt} = \gamma \frac{d}{dt}\left(\gamma c, \gamma \vec{u}\right)$$

$$\bar{A} = \gamma\left(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\vec{u} + \gamma\frac{d\vec{u}}{dt}\right)$$

$$\bar{A} = \gamma\left(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\vec{u} + \gamma\vec{a}\right)$$

Remark: in the instantaneous rest frame of the object we have $u = 0$ and $\gamma = 1$ thus $\bar{A} = (0, \vec{a})$ and thus $\bar{A} \cdot \bar{A} = 0 \iff \vec{a} \cdot \vec{a} = 0 \iff \vec{a} = 0$.

$$\boxed{\text{Th}^{m}/ \quad \bar{U} \cdot \bar{A} = 0}$$

Proof: left for homework, note $\bar{U} \cdot \bar{U} = -c^2$ as hint.

$$\boxed{\begin{array}{l}\text{Th}^{m}/ \text{ If } \bar{A} = (A^{\circ}, \vec{A}) \text{ then } A^{\circ} = \dfrac{\vec{u} \cdot \vec{A}}{c} \\ \text{and } \vec{A} = \gamma^2 \vec{a} + \gamma^4\left(\dfrac{\vec{u} \cdot \vec{a}}{c^2}\right)\vec{u}\end{array}}$$

relativistic acceleration

Proof: left for homework, need to derive identity for $\frac{d\gamma}{dt}$ to see this Th^{m}.

Remark: momentum is more important, focus there

4- Momentum and Relativistic Energy

Given a mass m with 4-velocity $\overline{U}$ we define,

Defn/ $\overline{P} = m\overline{U} = m\gamma(c, \vec{u}) = (m\gamma c, m\gamma\vec{u}) = (P^0, \vec{P})$

where $\gamma = \dfrac{1}{\sqrt{1 - u^2/c^2}}$ and $\vec{u} = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$

and we say $\vec{P} = m\gamma\vec{u}$ is the <u>relativistic</u> <u>momentum</u>

Let's study P^0 in the case $u^2/c^2 << 1$ (a.k.a. $u << c$)

$$\gamma(u) = \left(1 - u^2/c^2\right)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}\left(\tfrac{u^2}{c^2}\right) + \cdots \quad \text{(Binomial Series)}$$

$$m\gamma c^2 = mc^2\left(1 - \tfrac{1}{2}\tfrac{u^2}{c^2} + \cdots\right) = \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\tfrac{1}{2}mu^2 + \cdots}_{\text{kinetic energy}}$$

Defn/ $E = m\gamma c^2$ is the <u>relativistic</u> <u>energy</u> for a massive particle where $\gamma = \dfrac{1}{\sqrt{1 - u^2/c^2}}$

In retrospect, we see $\overline{P} = (E/c, \vec{P})$

$$\underbrace{\overline{P}\cdot\overline{P} = -E^2/c^2 + \vec{P}\cdot\vec{P} = (m\overline{U})\cdot(m\overline{U}) = m^2\,\overline{U}\cdot\overline{U} = -m^2c^2}_{\text{energy, momentum and rest mass } m \text{ are related and this is an invariant}}$$

<u>Remark</u>: if $m = 0$ then the formula above actually applies with appropriate interpretation. For light $P^2 = E^2/c^2$ that is $\boxed{E = PC}$ ← <u>LIGHT</u>.

Thm/ $E^2 = c^2P^2 + m^2c^4$ where $\vec{P} = m\gamma\vec{u}$ and $E = m\gamma c^2$

4-Momentum is conserved for isolated system

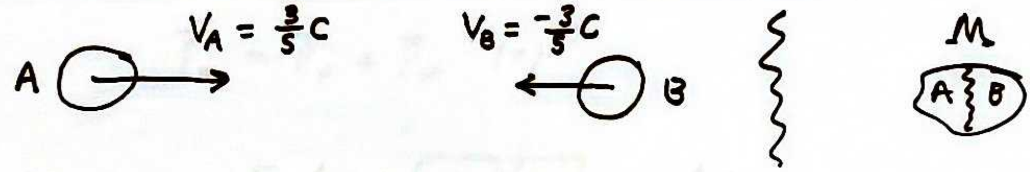
Consider some collection of objects, each with 4-momentum $P_{i,0}^{\mu}$ then if after some interaction the objects have 4-momentum $P_{i,f}^{\mu}$ then

$$\sum_{i=1}^{n} P_{i,0}^{\mu} = \sum_{i=1}^{n} P_{i,f}^{\mu}$$

$\mu = 0 \rightarrow$ conservation of relativistic energy

$\mu = 1,2,3 \rightarrow$ conservation of relativistic momentum

$\boxed{\text{E1}}$ Suppose two lumps of clay with mass m collide head-on at $\frac{3}{5}c$ then they stick together. What is resulting mass M of the composite lump?



$$\bar{P}_A = (m\gamma_A c, m\gamma_A V_A, 0, 0)$$

$$\bar{P} = (M\gamma c, M\gamma\vec{u})$$

$$\bar{P}_B = (m\gamma_B c, m\gamma_B V_B, 0, 0)$$

Notice $\gamma_A = \gamma_B$ since $\gamma(\frac{3}{5}c) = \gamma(-\frac{3}{5}c) = \dfrac{1}{\sqrt{1-\frac{9}{25}}} = \dfrac{1}{\sqrt{\frac{16}{25}}} = \dfrac{5}{4}$

Hence, $\bar{P}_A + \bar{P}_B = \bar{P}$ yields

$$0: \; m\gamma_A c + m\gamma_B c = 2m\gamma_A c = \underline{2m\left(\tfrac{5}{4}\right)c = M\gamma c}.$$

$$1: \; m\gamma_A\left(\tfrac{3}{5}c\right) + m\gamma_B\left(-\tfrac{3}{5}c\right) = 0 = M\gamma u_1$$

$$2: \; 0 = M\gamma u_2$$

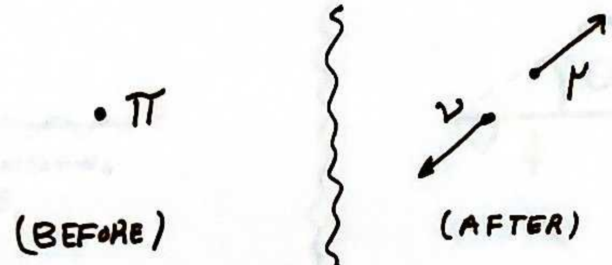
$$3: \; 0 = M\gamma u_3$$

We find $\vec{u} = 0$ thus $u = 0$ and $\gamma = 1$ consequently

$$\frac{5mc}{2} = Mc \quad \therefore \quad \boxed{M = \frac{5}{2}m}$$

$\boxed{E2}$ A pion at rest decays into a muon and a neutrino. Find the energy of outgoing muon in terms of the mass m_π of the pion and m_μ of the muon.

(this example from Griffiths, we assume $m_\nu = 0$ which is close to true...)



$$\bar{P}_\pi = (m_\pi c, \vec{0}) \quad \Big| \quad \bar{P}_\nu = (E_\nu/c, \vec{P}_\nu) \quad \bar{P}_\mu = (m_\mu \gamma c, m_\mu \gamma \vec{u})$$

$$\bar{P}_\pi = \bar{P}_\nu + \bar{P}_\mu \quad (*)$$

$$m_\pi c = E_\nu/c + \boxed{m_\mu \gamma c} \qquad \text{(zeroth component of } *)$$

$$\vec{0} = \vec{P}_\nu + m_\mu \gamma \vec{u} \qquad \text{(spatial components of } *)$$

Here $E_\nu = P_\nu c$ since $m_\nu = 0$

We wish to solve for $E_\mu = m_\mu \gamma c^2$.

$$\left\{ \begin{array}{c} E^2 - p^2c^2 = m^2c^4 \\ pc = \sqrt{E^2 - m^2c^4} \end{array} \right\}$$

$$\vec{P}_\mu = m_\mu \gamma \vec{u}$$

$$\underline{P_\mu = m_\mu \gamma u}.$$

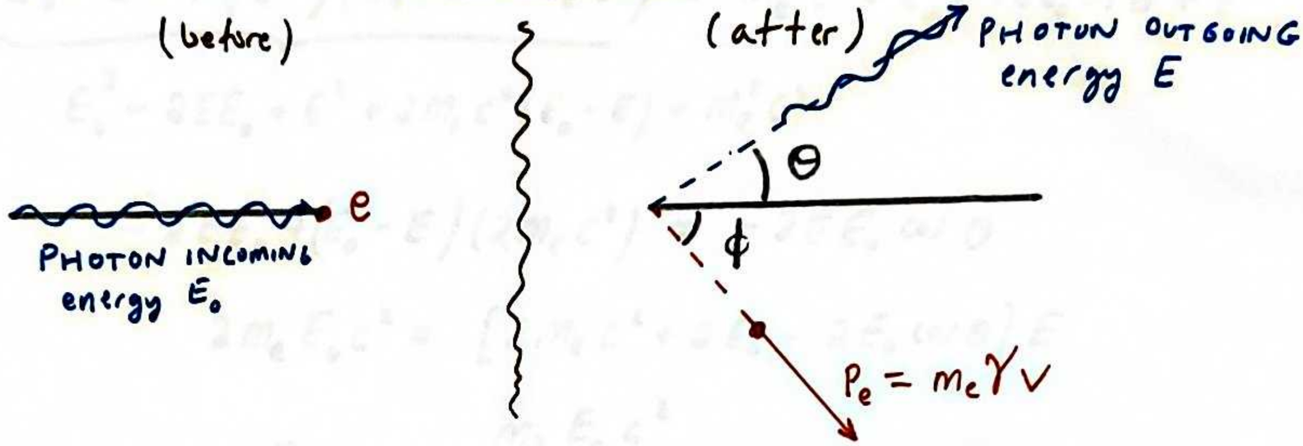
$$\begin{aligned} E_\mu = m_\mu \gamma c^2 &= m_\pi c^2 - E_\nu \\ &= m_\pi c^2 - c\,\|\vec{P}_\nu\| \\ &= m_\pi c^2 - m_\mu c \gamma u \\ &= m_\pi c^2 - \sqrt{E_\mu^2 - m_\mu^2 c^4} \end{aligned}$$

Then $(-E_\mu + m_\pi c^2)^2 = \left(\sqrt{E_\mu^2 - m_\mu^2 c^4}\right)^2 = E_\mu^2 - m_\mu^2 c^4$

$E_\mu^2 - 2m_\pi E_\mu c^2 + m_\pi^2 c^4 = E_\mu^2 - m_\mu^2 c^4 \quad \therefore \quad \boxed{E_\mu = \dfrac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}}$

E3 Compton Scattering

A photon of energy E_0 scatters off an electron at rest. Find the energy of the outgoing photon E as function of the scattering angle θ pictured below



$$\left(E_0/c + m_e c,\ E_0/c,\ 0\right) = \left(E/c + m_e \gamma c,\ \frac{E}{c}\cos\theta + P_e\cos\phi,\ \frac{E}{c}\sin\theta - P_e\sin\phi\right)$$

$$(*)\quad E_0 + m_e c^2 = E + m_e \gamma c^2 = E + \sqrt{m_e^2 c^4 + P_e^2 c^2}$$

$$E_0 = E\cos\theta + P_e\cos\phi$$

$$0 = \frac{E}{c}\sin\theta - P_e\sin\phi \Rightarrow \sin\phi = \frac{E\sin\theta}{P_e c}$$

$$\frac{E_0}{c} = \frac{E}{c}\cos\theta + P_e\sqrt{1 - \frac{E^2\sin^2\theta}{P_e^2 c^2}}$$

$$(E_0 - E\cos\theta)^2 = \left(P_e c\sqrt{1 - \left(\frac{E\sin\theta}{P_e c}\right)^2}\right)^2$$

$$E_0^2 - 2EE_0\cos\theta + E^2\cos^2\theta = P_e^2 c^2\left(1 - \frac{E^2\sin^2\theta}{P_e^2 c^2}\right)$$

$$= P_e^2 c^2 - E^2\sin^2\theta$$

$$\therefore\quad E_0^2 - 2EE_0\cos\theta + E^2 = P_e^2 c^2$$

~~Return~~ Return to $*$ (conservation of energy),

$$E_0 + m_e c^2 = E + \sqrt{m_e^2 c^4 + P_e^2 c^2}$$

$$= E + \sqrt{m_e^2 c^4 + E_0^2 - 2EE_0\cos\theta + E^2}$$

↴

E3 continued

$$E_0 + m_e c^2 - E = \sqrt{m_e^2 c^4 + E_0^2 - 2EE_0\cos\theta + E^2}$$

$$\underbrace{(E_0 - E + m_e c^2)(E_0 - E + m_e c^2)}_{E_0^2 - 2EE_0 + E^2 + 2m_e c^2(E_0 - E) + m_e^2 c^4} = m_e^2 c^4 + E_0^2 - 2EE_0\cos\theta + E^2$$

$$-2EE_0 + (E_0 - E)(2m_e c^2) = -2EE_0\cos\theta$$

$$2m_e E_0 c^2 = [2m_e c^2 + 2E_0 - 2E_0\cos\theta]E$$

$$E = \frac{m_e E_0 c^2}{m_e c^2 + E_0(1-\cos\theta)}$$

$$\boxed{E = \frac{1}{1/E_0 + (1-\cos\theta)/m_e c^2}}$$

In quantum mechanics we learn the energy of the photon is related to its frequency ν via $E = h\nu = \frac{hc}{\lambda}$ where h is Plank's constant

$$\frac{hc}{\lambda} = \frac{1}{\lambda_0/hc + (1-\cos\theta)/m_e c^2}$$

$$\boxed{\lambda = \lambda_0 + \frac{h}{m_e c}(1-\cos\theta)}$$

4-FORCE AND NEWTON'S LAW IN SPECIAL RELATIVITY

The 4-momentum leads to the 4-force,

Defn/ $k_i = \dfrac{dp_i}{dt} = \dfrac{d}{dt}(m\gamma u_i)$ is relativistic 3-force

$$K^\mu = \frac{dp^\mu}{d\tau} = \frac{d}{d\tau}(m\,U^\mu) = \gamma(u)\left(\frac{1}{c}\frac{dE}{dt},\ k_i\right)$$

We've used $\dfrac{dt}{d\tau} = \gamma(u)$ to convert $\dfrac{dp^\mu}{d\tau} = \dfrac{dt}{d\tau}\dfrac{dp^\mu}{dt} = \gamma(u)\dfrac{dp^\mu}{dt}$

since $(p^\mu) = (E/c,\ m\gamma\vec{u})$

<u>NEWTON'S LAW</u>: suppose $\vec{k} = \vec{F}$ is force on mass m_0 with velocity $\vec{u}$ then $\dfrac{d\vec{p}}{dt} = \dfrac{d}{dt}(m_0\gamma\vec{u}) = \vec{F}$

$$\vec{F} = \frac{d\vec{p}}{dt} = m_0\left(\frac{d\gamma}{dt}\vec{u} + \gamma\frac{d\vec{u}}{dt}\right)$$

However, $E = m_0\gamma c^2$ thus $\dfrac{dE}{dt} = m_0 c^2\dfrac{d\gamma}{dt} = \dfrac{dK}{dt} = \dfrac{d}{dt}\displaystyle\int \vec{F}\cdot d\vec{l}$

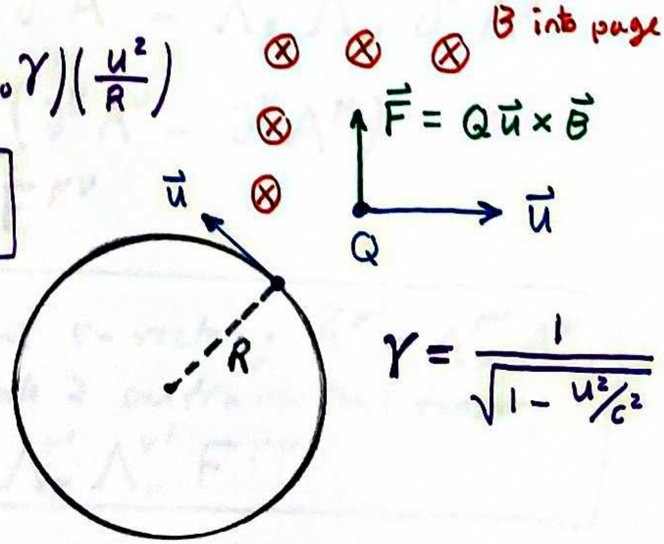
which gives $m_0 c^2 \dfrac{d\gamma}{dt} = \vec{F}\cdot\vec{u}$ and consequently,

$$\boxed{\vec{F} = m_0\gamma\vec{a} + \frac{1}{c^2}(F\cdot\vec{u})\vec{u}}$$

$\boxed{E4}$ A charge Q with velocity $\vec{u} \perp \vec{B}$ (uniform)

$$F = QuB = m_0\gamma a = (m_0\gamma)\left(\frac{u^2}{R}\right)$$

$$\therefore\ \boxed{R = \left(\frac{m_0 u}{QB}\right)\gamma}$$



$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

Remark: this differs from the Newtonian treatment of LECTURE 16 where we found $R = \dfrac{mV_0}{QB}$