LECTURE 3 : FUNCTIONS

Def^{n}/ Let X, Y be sets. A function from X into Y is a subset $f \subseteq X \times Y$ with the following properties

(a.) $\forall x \in X$, $\exists y \in Y$ such that $(x,y) \in f$

(b.) if $(x,y), (x,z) \in f$ then $y = z$ (f is single-valued)

We say $X = \text{domain}(f)$ and $Y = \text{codomain}(f)$ and write $f : X \to Y$.

Furthermore, $\text{range}(f) = \{ y \in Y \mid \exists x \in X \text{ such that } (x,y) \in f \}$

Def^{n}/ if $(x,y) \in f$ then we write $y = f(x)$ and say $f(x)$ is the <u>value of f at x</u>. Furthermore, if $y = f(x)$ then we say x is a <u>preimage of y under f</u>.

<u>Remark</u>: recall $\text{graph}(f) = \{ (x, f(x)) \mid x \in \text{dom}(f) \}$ for $f : \text{dom}(f) \to \mathbb{R}$ so our formal definition uses the graph of a function to define the meaning of "function". In truth, most practicing mathematicians don't really think this way, rather $f : X \to Y$ is thought of as a mapping from X into Y with the key property that if $f(x) = y$ and $f(x) = z$ then we must have $y = z$. Using $f \subseteq X \times Y$ is just a logical convenience to formalize the concept.

Def$^{\underline{n}}$/ Given functions $f : X \to Y$ and $g : X \to Y$ we say $f = g$ iff $f(x) = g(x)$ $\forall x \in X$. ⟵ Equality of Functions

When $A \subseteq X$ and $f : X \to Y$ is function we can make a new function $f|_A : A \to Y$ simply by using the same rule as f for the subset,

Def$^{\underline{n}}$/ Given function $f : X \to Y$ and $A \subseteq X$ we define the restriction of f to A by $f|_A : A \to Y$ where $(f|_A)(x) = f(x)$ for each $x \in A$

The next definition is <u>very</u> <u>important</u>.

codomain(f) = range(f)

Def$^{\underline{n}}$/ Let $f : X \to Y$ be a function,

(1.) f is <u>surjective</u> (onto) if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

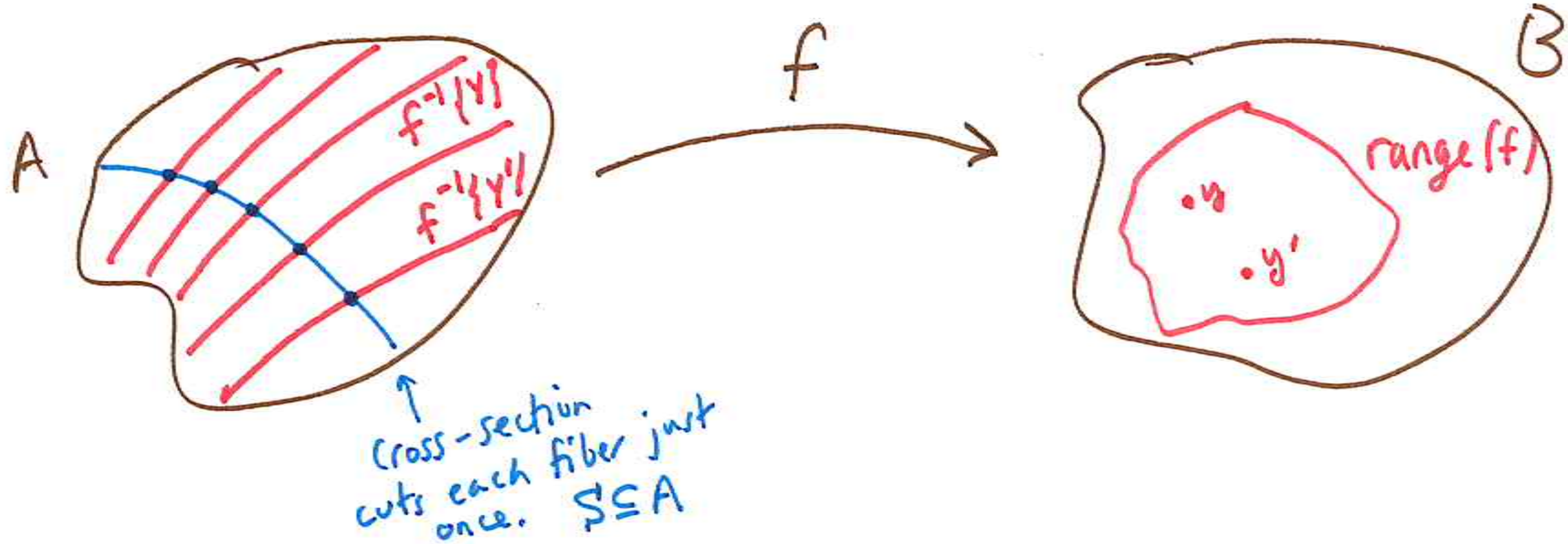
(2.) f is <u>injective</u> (1-to-1) if for every $x, x' \in X$ such that $f(x) = f(x') \Rightarrow x = x'$

Alternatively, for every $x, y \in X$ with $x \neq y$ we have $f(x) \neq f(y)$.

(3.) f is <u>bijective</u> iff f is both surjective and injective (1-1 and onto)

Bijections provide a one-to-one correspondence between X and Y.

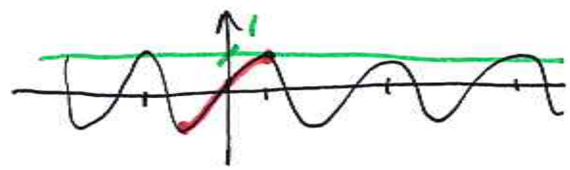
<u>Remark</u>: there is a way to take any $f : X \to Y$ and construct a corresponding bijection by suitably restricting both the domain and reducing the codomain.



$f: A \to B$ can make $\tilde{f}: A \to \text{range}(f)$ is onto range(f).

Then $\tilde{\tilde{f}}: S \to \text{range}(f)$ is 1-1 and onto ($\tilde{\tilde{f}}$ a bijection)

$\sin\left(\frac{\pi}{2}\right) = \sin\left(\frac{5\pi}{2}\right) = \sin\left(\frac{9\pi}{2}\right) \cdots$



Example: $f: \mathbb{R} \to \mathbb{R}$

$f(x) = \sin(x)$

$S = [-\pi/2, \pi/2]$

range$(f) = [-1, 1]$

$\tilde{f}: \mathbb{R} \to [-1, 1]$

$\tilde{f}(x) = \sin(x) \quad \forall x \in \mathbb{R}$

$\tilde{\tilde{f}}: [-\pi/2, \pi/2] \longrightarrow [-1, 1]$

$\tilde{\tilde{f}}(x) = \sin(x)$ for $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$

$\sin(x) = \sin(x')$

$x = x'$

$$\boxed{\begin{array}{l}\text{Th}^{\underline{m}}/\ (1.2.1 \text{ on pg. } 12)\ \text{Let } f: X \to Y.\ \text{If } \exists \text{ functions } g: Y \to X \text{ and } h: Y \to X \\ \text{such that } g(f(x)) = x \ \ \forall x \in X \text{ and } f(h(y)) = y \ \ \forall y \in Y \text{ then} \\ f \text{ is a bijection and } g = h = f^{-1}\end{array}}$$

<u>Proof</u>: Suppose f, g, h are given as in the statement of the Th$^{\underline{m}}$. We seek to show f onto. Let $y \in Y$ and notice $f(h(y)) = y$ where $h(y) \in X$ hence ~~[illegible]~~ $x = h(y)$ gives $f(x) = y$ thus f is surjective. Next we show f 1-to-1, suppose $f(x) = f(x')$ then $g(f(x)) = g(f(x')) \Rightarrow x = x'$. Thus f is a bijection since f is injective and surjective.

Recall $f^{-1}: Y \to X$ as a relation is simply given by swapping the entries in $f \subseteq X \times Y$; $(x, y) \in f \iff (y, x) \in f^{-1}$ which means $f^{-1}(y) = x$ iff $f(x) = y$. Notice f^{-1} of a bijection f is a function since f^{-1} containing both (y, x) and (y, x') means $f(x) = y$ and $f(x') = y$. Then $g(f(x)) = g(y) \Rightarrow x = g(y)$ & $g(f(x')) = g(y) \Rightarrow x' = g(y)$ thus f^{-1} is single-valued and is therefore a <u>function</u>. Finally, to see $g = h = f^{-1}$ consider,

$$g(f(x)) = g(y) = x \Rightarrow g(y) = f^{-1}(y) \quad \forall y \in Y.$$

$$f(h(y)) = y \Rightarrow h(y) = f^{-1}(y) \ \ \forall y \in Y. \text{ Hence } f^{-1} = h = g. //$$

Example 1:

$f : [1,2] \longrightarrow [2,5]$

$f(x) = x^2+1$

$y = x^2+1 \Rightarrow x^2 = y-1 \Rightarrow x = \pm\sqrt{y-1}$, however $1 \le x \le 2$ so choose $+$

$\Rightarrow f^{-1}(y) = \sqrt{y-1}$ (this is probably the inverse function of f)

- Easy to prove onto,

  Let $y \in [2,5]$ then calculate $2 \le y \le 5 \Rightarrow 1 \le y-1 \le 4 \Rightarrow 1 \le \sqrt{y-1} \le 2$

  thus $\sqrt{y-1} \in [1,2]$ and $f(\sqrt{y-1}) = (\sqrt{y-1})^2+1 = y-1+1 = y$ $\therefore$ f is surjective.

- Likewise 1-1,

  Suppose $f(x) = f(x')$ for $x, x' \in [1,2]$ then $x^2+1 = (x')^2+1 \Rightarrow x^2 = (x')^2$

  thus $x^2 - (x')^2 = (x-x')(x+x') = 0$ therefore $x = \pm x'$ but $1 \le x, x' \le 2$

  therefore $x = x'$ and we've shown f is injective.

Example 2:

$f : \mathbb{R} \longrightarrow \mathbb{R}$

$f(x) = 3x - 8$

$3x - 8 = y$
$x = \frac{y+8}{3}$

ONTO? Let $y \in \mathbb{R}$ and consider $x = \frac{1}{3}(y+8) \in \mathbb{R}$

then $f\left(\frac{1}{3}(y+8)\right) = 3\left(\frac{1}{3}(y+8)\right) - 8 = y+8-8 = y$.

Therefore f is onto $\mathbb{R}$.

1-1? Suppose $x, x' \in \mathbb{R}$ and $f(x) = f(x')$ then

$3x - 8 = 3x' - 8 \Rightarrow 3x = 3x' \Rightarrow x = x'$ $\therefore$ f is 1-1.

Remark: for $f: X \to Y$ a function if $f^{-1}: Y \to X$ is known to be a function as well then both f and f^{-1} are bijections.

Let $y \in Y$ then $x = f^{-1}(y) \in X$ and $f(x) = f(f^{-1}(y)) = y$ $\therefore$ f is onto. Y.
Suppose $f(a) = f(b)$ then $f^{-1}(f(a)) = f^{-1}(f(b)) \Rightarrow a = b$ $\therefore$ f is 1-1.
Thus f is a bijection since it is surjective and injective.

Let $x \in X$ then $y = f(x) \in Y$ and $f^{-1}(y) = f^{-1}(f(x)) = x$ $\therefore$ f^{-1} is onto X.
Suppose $f^{-1}(y) = f^{-1}(y')$ then $f(f^{-1}(y)) = f(f^{-1}(y')) \Rightarrow y = y'$ $\therefore$ f^{-1} is 1-1.
Thus f^{-1} is a bijection.

Defn/ Let $f: X \to Y$ and suppose $A \subseteq X$ and $B \subseteq Y$ then

$$f(A) = \{ f(x) \mid x \in A \} = \{ y \in Y \mid \exists x \in A \text{ such that } f(x) = y \}$$

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \}$$

We say $f(A)$ is the image of A under f
and we say $f^{-1}(B)$ is the inverse image of B under f
If $y \in Y$ then $f^{-1}\{y\} = \{ x \in X \mid f(x) = y \}$ is the fiber of f under y.

Example: $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(x)$

$f(\pi\mathbb{Z}) = \{0\}$ since $\pi\mathbb{Z} = \{ \pi z \mid z \in \mathbb{Z} \}$ and $\sin(\pi z) = 0$

$f^{-1}\{1\} = \{ \frac{\pi}{2} + 2\pi n \mid n \in \mathbb{Z} \}$

$f^{-1}\{2\} = \emptyset$ $\quad$ $2 \notin \text{range}(f) = [-1, 1]$

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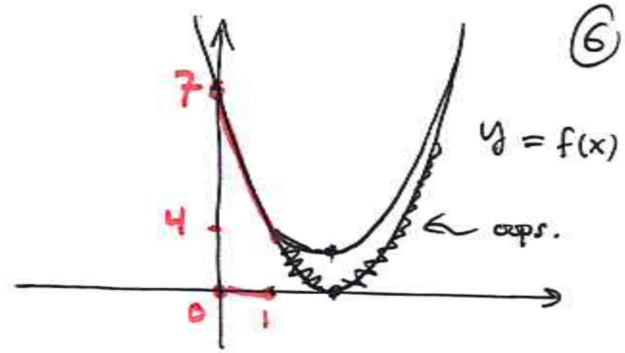
Example: Let $f(x) = 3 + (x-2)^2$



$$f([0,1]) = \{ f(x) \mid x \in [0,1] \}$$
$$= \{ 3 + (x-2)^2 \mid 0 \le x \le 1 \}$$
$$= [4, 7] \quad \text{(from the graph)}$$

range$(f) = [3, \infty)$

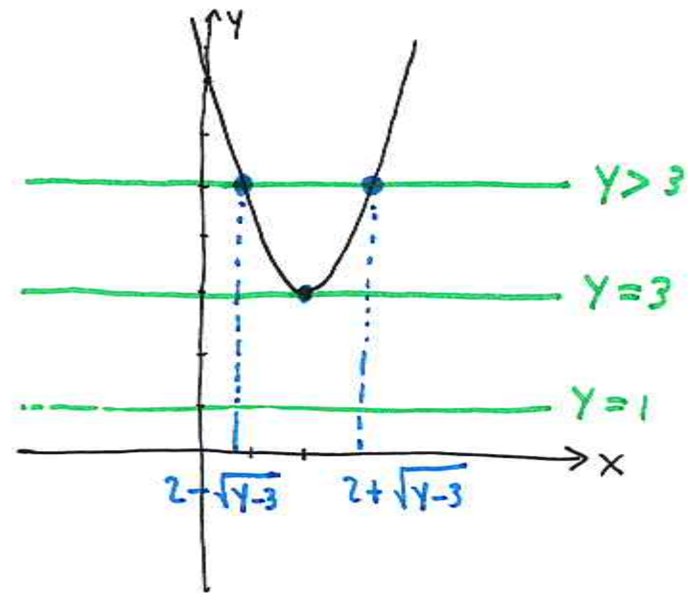
$$f^{-1}\{1\} = \{ x \mid f(x) = 1 \}$$
$$= \{ x \mid 3 + (x-2)^2 = 1 \}$$
$$= \emptyset$$

$$\underbrace{(x-2)^2 = -2}_{\text{No sol'ns in } \mathbb{R}}$$

$$f^{-1}\{3\} = \{ x \mid 3 + (x-2)^2 = 3 \} = \{2\}$$

$\underline{y > 3}$

$$f^{-1}\{y\} = \{ x \mid 3 + (x-2)^2 = y \}$$
$$= \{ x \mid (x-2)^2 = y - 3 \}$$
$$= \{ x \mid x - 2 = \pm\sqrt{y-3} \}$$
$$= \{ x \mid x = 2 \pm \sqrt{y-3} \}$$
$$= \{ 2 + \sqrt{y-3},\ 2 - \sqrt{y-3} \}$$



· fibers of f are either empty, have just one pt. or contain a pair of preimages.

Th$^{\underline{m}}$ (1.2.3) Let $f : X \to Y$ be fnct, $A \subseteq X$ and $B \subseteq Y$. Then

(a.) $A \subseteq f^{-1}(f(A))$

(b.) $f(f^{-1}(B)) \subseteq B$

<u>Proof (a)</u> Let $x \in A$ then $f(x) \in f(A)$. Thus $x \in f^{-1}(f(A))$ by def$^{\underline{n}}$ of inverse image. Hence $A \subseteq f^{-1}(f(A))$. //

<u>Proof (b)</u> : part of your hwk ☺

Th$^{\underline{m}}$/(1.2.4) Let $f : X \to Y$ be fnct, Let $A, B \subseteq X$ and $C, D \subseteq Y$. Then

(a.) if $C \subseteq D$ then $f^{-1}(C) \subseteq f^{-1}(D)$

(b.) $f^{-1}(D - C) = f^{-1}(D) - f^{-1}(C)$

(c.) if $A \subseteq B$ then $f(A) \subseteq f(B)$

(d.) $f(A - B) \supseteq f(A) - f(B)$

$\supseteq$ ← contains

$f(A) - f(B) \subseteq f(A - B)$

<u>Proof</u>: (b.) is given in text. I'll do (a.)
Suppose $C \subseteq D$. Let $x \in f^{-1}(C)$ then $f(x) \in C$ by def$^{\underline{n}}$ of inverse image. Hence $f(x) \in C \subseteq D \Rightarrow f(x) \in D$ thus $x \in f^{-1}(D)$ which shows $f^{-1}(C) \subseteq f^{-1}(D)$. //

(d.) Let $x \in f(A) - f(B)$ then $\exists\, a \in A$ s.t. $f(a) = x$ and $\nexists\, b \in B$ s.t. $f(b) = x$. This implies $a \notin B$ hence $a \in A - B$ and $x = f(a)$ so $x \in f(A - B)$. Therefore, $f(A) - f(B) \subseteq f(A - B)$. //

$\underline{\text{Th}^{\underline{m}}\ (1.2.5)}$ Let $f : X \to Y$ be fnct and $\{A_\alpha\}_{\alpha \in I}$ an indexed family of subsets of X and $\{B_\beta\}_{\beta \in J}$ an indexed family of subsets of Y. Then,

(a.) $f(\cup A_\alpha) = \cup f(A_\alpha)$

(b.) $f(\cap A_\alpha) \subseteq \cap f(A_\alpha)$

(c.) $f^{-1}(\cup B_\beta) = \cup f^{-1}(B_\beta)$

(d.) $f^{-1}(\cap B_\beta) = \cap f^{-1}(B_\beta)$

$$f\Big(\bigcup_{\alpha \in I} A_\alpha\Big) = \bigcup_{\alpha \in I} f(A_\alpha)$$

<u>PROOF (c.)</u>

Let $x \in f^{-1}(\cup B_\beta)$ then $f(x) \in \bigcup_{\beta \in J} B_\beta$ thus $\exists \beta_0 \in J$ for which $f(x) \in B_{\beta_0}$. Thus $x \in f^{-1}(B_{\beta_0})$ by def$^{\underline{n}}$ of preimage. Hence $x \in \cup f^{-1}(B_\beta)$ by def$^{\underline{n}}$ of union. We've shown $f^{-1}(\cup B_\beta) \subseteq \cup f^{-1}(B_\beta)$.

Let $x \in \cup f^{-1}(B_\beta)$ then $\exists \beta_1 \in J$ for which $x \in f^{-1}(B_{\beta_1})$ hence $f(x) \in B_{\beta_1} \subseteq \cup B_\beta$ thus $x \in f^{-1}(\cup B_\beta)$ and so $\cup f^{-1}(B_\beta) \subseteq f^{-1}(\cup B_\beta)$. Therefore, by double-containment, we've shown $f^{-1}(\cup B_\beta) = \cup f^{-1}(B_\beta)$ //

Def^n/ Given fncts $f : X \to Y$ and $g : Y \to Z$ we define $g \circ f : X \to Z$ by $(g \circ f)(x) = g(f(x))\ \forall x \in X$.

Th^m/ (1.2.6) Let $f : X \to Y$, $g : Y \to Z$ be fncts and suppose $B \subseteq Z$. Then,

(a.) $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$

(b.) if f and g are injective then $g \circ f$ is injective

(c.) if f and g are surjective then $g \circ f$ is surjective

(d.) if $g \circ f$ is injective then f is injective

(e.) if $g \circ f$ is surjective then g is surjective

SOCKS SHOES FOR INVERSE IMAGES. NOTICE f, g need not be invertible as fncts. for this to hold.

So... the composition of bijections is a bijection.

Proof: text does (d.), I'll do (b.) & (e.)

(b.) Suppose f and g are 1-1. Consider $(g \circ f)(x) = (g \circ f)(x')$ then $g(f(x)) = g(f(x')) \Rightarrow f(x) = f(x')$ since g 1-1 and $f(x) = f(x') \Rightarrow x = x'$ since f is 1-1. Thus $g \circ f$ is likewise 1-1 as $(g \circ f)(x) = (g \circ f)(x') \Rightarrow x = x'$. //

(e.) Suppose $g \circ f$ is onto. Let $z \in Z$ then $\exists x \in X$ such that $(g \circ f)(x) = z$. Thus $g(f(x)) = z$ and as $f(x) \in Y$ we find $\exists y = f(x) \in Y$ such that $g(y) = z$ $\therefore$ g is ONTO Z. //

$$\left(g : Y \to Z, \quad y \xrightarrow{g} z \right)$$