LECTURE 3: INDEX NOTATION FOR $\mathbb{R}^3$ or $\mathbb{R}^4$ and more...

Convention: i, j, k etc.. run from $1, 2, 3$ (Latin Indices)

μ, ν, α, β etc.. run from $0, 1, 2, 3$ (Greek Indices)

Convention: repeated indices are summed over their values (Einstein's summation convention)

VECTOR ALGEBRA FOR CALCULUS III

Kroeneker Delta

$$\vec{A} \cdot \vec{B} = A^1 B^1 + A^2 B^2 + A^3 B^3 = \sum_{i=1}^{3} A^i B^i = A^i B^i = \delta_{ij} A^i B^j \quad \text{where } \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$\hat{x}_1 = \langle 1, 0, 0 \rangle$

$\hat{x}_2 = \langle 0, 1, 0 \rangle$

$\hat{x}_3 = \langle 0, 0, 1 \rangle$

$$\vec{A} \times \vec{B} = (A_2 B_3 - A_3 B_2)\hat{x}_1 + (A_3 B_1 - A_1 B_3)\hat{x}_2 + (A_1 B_2 - A_2 B_1)\hat{x}_3$$

$$\vec{A} \times \vec{B} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} A_i B_j \hat{x}_k = \epsilon_{ijk} A^i B^j \hat{x}_k = \epsilon_{abc} A^a B^b \hat{x}_c$$

Def^n/ Levi-Civita symbol in $n=3$

$$\boxed{\begin{aligned} &\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ &\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \\ &\epsilon_{ijk} = 0 \quad \text{whenever} \\ &\text{one or } \cancel{\text{two}} \text{ of } i,j,k \text{ are repeated} \\ &\epsilon_{11\cancel{2}} = \epsilon_{232} = \epsilon_{333} = 0 \text{ etc..} \end{aligned}}$$

$$\left.\begin{aligned} \hat{x}_i \cdot \hat{x}_j &= \delta_{ij} \\ \hat{x}_i \times \hat{x}_j &= \epsilon_{ijk} \hat{x}_k \end{aligned}\right\} \quad i, j \text{ are } \underline{\text{free}} \text{ indices in these eq}^{\text{n}}\text{s.}$$

<u>Comment</u>: up/down indices are same for Euclidean $\mathbb{R}^3$; $A^i = A_i$, $\delta_{ij} = \delta^{ij} = \delta^i{}_j = \delta_i{}^j$ however, we'll soon see there is a difference for Minkowski Space.

$$\boxed{\text{Th}^{\underline{m}}/\quad \epsilon_{ijk}\,\epsilon_{kab} = \delta_{ia}\,\delta_{jb} - \delta_{ib}\,\delta_{ja}}$$

← RHS/LHS both antisymmetric in i,j and a,b index pairs and LHS = RHS in case
$\epsilon_{12k}\,\epsilon_{k12} = \epsilon_{123}\,\epsilon_{312} = 1$ (LHS)
$\delta_{11}\delta_{22} - \delta_{12}\delta_{21} = 1$ (RHS)

The result above yields a nontrivial result in 3D-vector algebra:

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \left(\sum_{i,j,k}\epsilon_{ijk}\,A_i B_j\,\hat{x}_k\right)\cdot\left(\sum_{a,b,l}\epsilon_{abl}\,C_a D_b\,\hat{x}_l\right)$$

$$= \sum_{i,j,k,l,a,b}\epsilon_{ijk}\,\epsilon_{abl}\,A_i\,B_j\,C_a\,D_b\,\underbrace{\hat{x}_k\cdot\hat{x}_l}_{\delta_{kl}}$$

$$= \sum_{i,j,a,b}A_i\,B_j\,C_a\,D_b\sum_k\sum_l\epsilon_{ijk}\,\epsilon_{abl}\,\delta_{kl}$$

$\epsilon_{abk} = -\epsilon_{akb} = -(-\epsilon_{kab})$

$$= \sum_{i,j,a,b}A_i\,B_j\,C_a\,D_b\sum_k\epsilon_{ijk}\,\epsilon_{kab}$$

Sorry, forgot to omit Σ's

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = A_i\,B_j\,C_a\,D_b\,\epsilon_{ijk}\,\epsilon_{kab}$$

$$= A_i\,B_j\,C_a\,D_b\,(\delta_{ia}\,\delta_{jb} - \delta_{ib}\,\delta_{ja})$$

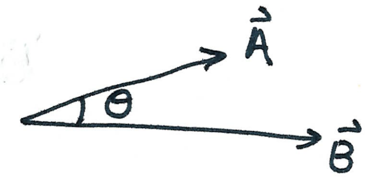
$$= A_i\,B_j\,C_a\,D_b\,\underbrace{\delta_{ia}\,\delta_{jb}}_{\substack{\text{set } i=a\\ \text{set } j=b}} - A_i\,B_j\,C_a\,D_b\,\underbrace{\delta_{ib}\,\delta_{ja}}_{\substack{\text{set } a=b\\ \text{set } j=a}}$$

$$= A_i\,B_j\,C_i\,D_j - A_i\,B_j\,C_j\,D_i$$

$$= (A_i\,C_i)(B_j\,D_j) - (A_i\,D_i)(B_j\,C_j) = \underline{(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})}\ //$$

We derived $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

the Law of Cosines is essentially equivalent to



$$\vec{A}\cdot\vec{B} = \|\vec{A}\|\,\|\vec{B}\|\cos\theta$$

where $\|\vec{A}\| = \sqrt{\vec{A}\cdot\vec{A}}$ is the length of $\vec{A}$. Note then

$$\begin{aligned}
\|\vec{A}\times\vec{B}\|^2 &= (\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) \\
&= (\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B}) - (\vec{A}\cdot\vec{B})(\vec{B}\cdot\vec{A}) \\
&= \|\vec{A}\|^2\,\|\vec{B}\|^2 - (\|\vec{A}\|\,\|\vec{B}\|\cos\theta)^2 \\
&= \|\vec{A}\|^2\,\|\vec{B}\|^2\,(1-\cos^2\theta) \\
&= (\|\vec{A}\|\,\|\vec{B}\|\sin\theta)^2 \longrightarrow \boxed{\|\vec{A}\times\vec{B}\| = \|\vec{A}\|\,\|\vec{B}\|\sin\theta}
\end{aligned}$$

Remark: index notation is also important and useful for matrix algebra

let $i,j = 1,2,\dots,n$ and A, B be $(n\times n)$-matrices

$$(AB)_{ij} = A_{ik}B_{kj}$$

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$(cA)_{ij} = cA_{ij}$$

$\epsilon_{12\cdots n} = 1$, $\epsilon_{i_1 i_2\cdots i_n}$ completely antisymmetric

$$\det(A) = \epsilon_{i_1 i_2\cdots i_n}\underbrace{A_{i_1 1}A_{i_2 2}\cdots A_{i_n n}}^{\text{multilinear in columns of } A}$$

Block multiplication

$$(A_{I,J}) = \left(\begin{array}{c|c} A_{ij} & A_{i\nu} \\ \hline A_{\mu j} & A_{\mu\nu} \end{array}\right)$$

$i,j = 1,2,\dots,n_1$

$\mu,\nu = 1,2,\dots,n_2$

$$A_{iJ}B_{Jk} = A_{ij}B_{jk} + A_{i\nu}B_{\nu k} = (AB)_{ik}$$

$$A_{\alpha J}B_{J\beta} = A_{\alpha j}B_{j\beta} + A_{\alpha\nu}B_{\nu\beta} = (AB)_{\alpha\beta}$$

Remark on matrix algebra continued

$$A_{\alpha J} B_{Jk} = A_{\alpha j} B_{jk} + A_{\alpha \nu} B_{\nu k} = (AB)_{\alpha k} \qquad \alpha, \beta, \nu = 1, 2, \ldots, n_2$$

$$A_{\kappa J} B_{J\beta} = A_{\kappa j} B_{j\beta} + A_{\kappa \nu} B_{\nu\beta} = (AB)_{\kappa\beta} \qquad i, j, k = 1, 2, \ldots, n_1$$

$$\left(A_{IJ}\right) = \left(\begin{array}{c|c} A_{ij} & A_{i\nu} \\ \hline A_{\alpha j} & A_{\alpha\nu} \end{array}\right) = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \quad \text{likewise} \quad \left(B_{IJ}\right) = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$(AB) = \left[\begin{array}{c|c} (AB)_{ij} & (AB)_{i\nu} \\ \hline (AB)_{\alpha j} & (AB)_{\alpha\nu} \end{array}\right] = \left[\begin{array}{c|c} A_{ik} B_{kj} + A_{i\beta} B_{\beta j} & A_{ik} B_{k\nu} + A_{i\beta} B_{\beta\nu} \\ \hline A_{\alpha k} B_{kj} + A_{\alpha\beta} B_{\beta j} & A_{\alpha k} B_{k\nu} + A_{\alpha\beta} B_{\beta\nu} \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_{11} B_{11} + A_{12} B_{21} & A_{11} B_{12} + A_{12} B_{22} \\ \hline A_{21} B_{11} + A_{22} B_{21} & A_{21} B_{12} + A_{22} B_{22} \end{array}\right]$$

Application to Block Diagonal Case

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_n = \begin{bmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & & & \boxed{A_n} \end{bmatrix}$$ where $A_1, A_2, \ldots, A_n$ are square matrices

$$AB = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{bmatrix} \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_n \end{bmatrix} = \begin{bmatrix} A_1 B_1 & & & \\ & A_2 B_2 & & \\ & & \ddots & \\ & & & A_n B_n \end{bmatrix}$$

SYMMETRIC vs. ANTISYMMETRIC

Def^{n}/ $T_{i_1 i_2 \cdots i_k}$ is <u>symmetric in i_a i_b</u> if exchanging $i_a \leftrightarrow i_b$ does not change the value; $T_{i_1 \cdots i_a \cdots i_b \cdots i_k} = T_{i_1 \cdots i_b \cdots i_a \cdots i_k}$

$T_{i_1 i_2 \cdots i_k}$ is <u>antisymmetric in i_a, i_b</u> if exchanging $i_a \leftrightarrow i_b$ flips the sign; $T_{i_1 \cdots i_a \cdots i_b \cdots i_k} = -T_{i_1 \cdots i_b \cdots i_a \cdots i_k}$

If $T_{i_1 \cdots i_k}$ is symmetric in all index pairs then T_I is <u>completely symmetric</u>

If $T_{i_1 \cdots i_k}$ is antisymmetric in all index pairs then T_I is <u>completely antisymmetric</u>

<u>Remark</u>: if S_n is set of permutations on $\{1, 2, .., n\}$ and $sgn(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ even} \\ -1 & \text{if } \sigma \text{ odd} \end{cases}$

then $T_I = T_{\sigma(I)}$ $\forall$ multindices when T symmetric

and $T_{\sigma(I)} = sgn(\sigma)\, T_I$ $\forall$ multindices when T is antisymmetric.

Def^{n}/ $T_{(i_1 i_2 \cdots i_p)} = \frac{1}{p!} \sum_{\sigma} T_{\sigma(i_1, \dots, i_p)}$ (sum over all permutations of $i_1, .., i_p$)

$T_{[i_1, \dots, i_p]} = \frac{1}{p!} \sum_{\sigma} sgn(\sigma)\, T_{\sigma(i_1, .., i_p)}$ (sum over all permutations of $i_1, \dots, i_p$)

$$T_{(ij)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)$$

$$T_{[ij]} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

$$T_{(i_1 i_2 i_3)} = \frac{1}{6}\left(T_{i_1 i_2 i_3} + T_{i_2 i_3 i_1} + T_{i_3 i_1 i_2} + T_{i_3 i_2 i_1} + T_{i_2 i_1 i_3} + T_{i_1 i_3 i_2}\right)$$

$$T_{[i_1 i_2 i_3]} = \frac{1}{6}\left(T_{i_1 i_2 i_3} + T_{i_2 i_3 i_1} + T_{i_3 i_1 i_2} - T_{i_3 i_2 i_1} - T_{i_2 i_1 i_3} - T_{i_1 i_3 i_2}\right)$$

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IDENTITY: $T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}) = T_{(ij)} + T_{[ij]}$ only for pair of indices, does not extend to three or more.

Th^m/ If we <u>contract</u> (sum over) a pair of symmetric and antisymmetric indices then the sum is zero; $T_{ij} = T_{ji}$ and $A_{ij} = -A_{ji}$ $\forall i,j$ then $\sum_{i,j} A_{ij} T_{ij} = 0$ (a.k.a. $A_{ij} T_{ij} = 0$ in our concise notation)

<u>Proof</u>: I'll suspend the summation convention for this proof. Let $A_{ij} = -A_{ji}$ and $T_{ij} = T_{ji}$ $\forall i,j$. Then $A_{ii} = -A_{ii} \Rightarrow A_{ii} = 0$. Consider,

$$\sum_{i,j} A_{ij} T_{ij} = \underbrace{\sum_{i<j} A_{ij} T_{ij}}_{\substack{\text{let } k=i \\ \text{let } j=l}} + \underbrace{\cancel{\sum_{i=j} A_{ij} T_{ij}}}_{\text{as } A_{ii}=0} \xrightarrow{} 0 + \underbrace{\sum_{i>j} A_{ij} T_{ij}}_{\substack{\text{let } k=j \\ \text{let } l=i}}$$

$$= \sum_{k<l} A_{kl} T_{kl} + \sum_{l>k} A_{lk} T_{lk}$$

$$= \sum_{k<l} (A_{kl} T_{kl} + A_{lk} T_{lk})$$

$$= \sum_{k<l} (\cancel{A_{kl} T_{kl}} - \cancel{A_{kl} T_{kl}})$$

$$= 0 \,//$$

Remark: A or T can carry additional indices, this identity still holds.

<u>Application</u>: if $col_i(A) = col_j(A)$ then $\det(A) = \underbrace{\epsilon_{i_1 \cdots i \cdots j \cdots i_n}}_{\text{antisymmetric in } i \,\&\, j} \underbrace{A_{i_1 1} \cdots A_{i_i i} \cdots A_{i_j j} \cdots A_{i_n n}}_{\text{symmetric in } i,j} = 0$