Lecture 4: Lorentz Transformations & The Poincare Group

The Poincare Group includes <u>Lorentz Transformations</u> and <u>Spacetime Translations</u>. These transformations leave $(\Delta s)^2 = \eta_{\mu\nu}(\Delta x^\mu)(\Delta x^\nu)$ unchanged.

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Def$^{\underline{n}}$/ $x^{\mu'} = x^\mu + a^\mu$ defines a <u>spacetime translation</u> (a.k.a. $x' = x + a$)

$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^\nu$ defines a <u>Lorentz Transformation</u> provided $\eta = \Lambda^T \eta \Lambda$ $\underbrace{(x' = \Lambda x)}_{\text{matrix notation}}$

- Event x_1 and x_2 <u>translate</u> to $x_1' = x_1 + a$ and $x_2' = x_2 + a$ then,

$$\boxed{\Delta x' = x_2' - x_1' = (x_2 + a) - (x_1 + a) = x_2 - x_1 = \Delta x}$$

thus $\Delta x' = \Delta x$ and we find the increment between events is invariant under spacetime translations $\therefore (\Delta s')^2 = \eta_{\mu\nu}(\Delta x')^\mu (\Delta x')^\nu = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = (\Delta s)^2$.

- Events x_1 and x_2 are <u>Lorentz transformed</u> to $x_1' = \Lambda x_1$ and $x_2' = \Lambda x_2$

thus $\Delta x' = x_2' - x_1' = \Lambda x_2 - \Lambda x_1 = \Lambda(x_2 - x_1) = \Lambda \Delta x$

$$\boxed{\begin{aligned} (\Delta s')^2 &= (\Delta x')^T \eta (\Delta x') \\ &= (\Lambda \Delta x)^T \eta (\Lambda \Delta x) \quad \text{Socks-shoes identity for transpose of matrix} \\ &= (\Delta x)^T \Lambda^T \eta \Lambda \Delta x \quad \eta = \Lambda^T \eta \Lambda \\ &= (\Delta x)^T \eta \Delta x \\ &= (\Delta s)^2 \end{aligned}}$$

Lorentz Transformations:

$$\boxed{\text{Def}^{n}/\ O(3,1) = \{\Lambda \in \mathbb{R}^{4\times 4} \mid \eta = \Lambda^T \eta \Lambda\}}$$

$\mathbb{R}^4 = \text{span}\{e_0, e_1, e_2, e_3\}$

$\Theta^\mu(e_\nu) = \delta^\mu_\nu$

$\Theta^\mu : \mathbb{R}^4 \to \mathbb{R}$

$$\Lambda \in O(3,1) \Rightarrow \eta_{\sigma\rho} = \Lambda^{\mu'}{}_\sigma \eta_{\mu'\nu'} \Lambda^{\nu'}{}_\rho = \Lambda^{\mu'}{}_\sigma \Lambda^{\nu'}{}_\rho \eta_{\mu'\nu'}$$

If $g : M \times M \to \mathbb{R}$ is the Minkowski Metric then $g = \eta_{\mu\nu}\, \Theta^\mu \otimes \Theta^\nu$

Setting $\Lambda^{\mu'}{}_\sigma \Theta^\sigma = \Theta^{\mu'}$ note $\eta_{\mu'\nu'} \Theta^{\mu'} \otimes \Theta^{\nu'} = \eta_{\mu'\nu'} \Lambda^{\mu'}{}_\sigma \Lambda^{\nu'}{}_\rho\, \Theta^\sigma \otimes \Theta^\rho = \eta_{\sigma\rho}\, \Theta^\sigma \otimes \Theta^\rho$

The matrix of the Minkowski Metric is η with respect to any inertial frame

Remark: in $\mathbb{R}^3$ the matrix of the dot-product is the identity if we use any coordinate system which is rotated from the usual CARTESIAN COORD. SYSTEM

$$g(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w} = v^T w = v^T I w \ \ \& \ \ g(R\vec{v}, R\vec{w}) = (Rv)^T R w = v^T \underbrace{R^T R}_{I} w = v^T w \text{ for } R \in O(3)$$

Time Reversal

$$\Lambda_T = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

PARITY (spatial inversions)

$$P_1 = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \quad P_3 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

$\underbrace{\Lambda_T,\ P_1,\ P_2,\ P_3}_{\text{like reflections for } \mathbb{R}^3}$ $\in O(3,1)$. Check it $\Lambda_T^T \eta \Lambda_T = \eta^3 = \begin{pmatrix} -1^3 & & & \\ & 1^3 & & \\ & & 1^3 & \\ & & & 1^3 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta.$

these have determinant -1.

$$P_1^T \eta P_1 = \begin{bmatrix} 1(-1)1 & & & \\ & (-1)(1)(-1) & & \\ & & (1)(1)(1) & \\ & & & (1)(1)(1) \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \eta$$

Th^m/ If $\Lambda \in O(3,1)$ then $\det(\Lambda) = \pm 1$

Proof: $\Lambda \in O(3,1) \Rightarrow \eta = \Lambda^T \eta \Lambda$

$$\therefore \det(\eta) = \det(\Lambda^T \eta \Lambda) = \det(\Lambda^T)\det(\eta)\det(\Lambda)$$

$$\Rightarrow 1 = \det(\Lambda)\det(\Lambda)$$

$$\therefore \underline{\det(\Lambda) = \pm 1}.$$

Def^n/ $SO(3,1) = \{\Lambda \in O(3,1) \mid \det(\Lambda) = 1\}$

$SO(3,1)^{\uparrow} = \{\Lambda \in SO(3,1) \mid {\Lambda^{0'}}_0 \geq 1\}$ ← proper orthochronous Lorentz group.

EXAMPLES

1.) $\Lambda = \left[\begin{array}{c|c} 1 & \\ \hline & R \end{array}\right]$ where $R \in SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^TR = I,\ \det R = 1\}$ ← rotations on $\mathbb{R}^3$

you can check $\Lambda^T \eta \Lambda = \left[\begin{array}{c|c} -1 & \\ \hline & R^TR \end{array}\right] = \left[\begin{array}{c|c} -1 & \\ \hline & I_3 \end{array}\right] = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \eta$

and $\det(\Lambda) = \det(1)\det(R) = 1$ thus $\Lambda \in SO(1,3)$. (spatial rotation in space time)

2.) $\Lambda_1 = \left[\begin{array}{cc|cc} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ has $\det\Lambda_1 = \det\begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{bmatrix} \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= (\cosh^2\phi - \sinh^2\phi)(1)$$

$$= 1$$

also, $\Lambda_1^T \eta \Lambda_1 = \eta \therefore \Lambda_1 \in SO(1,3)$.

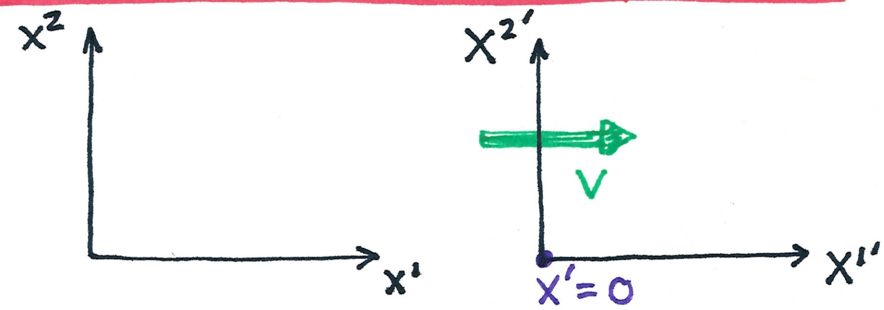
Similarly, $\Lambda_2 = \begin{bmatrix} \cosh\phi & 0 & -\sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ & $\Lambda_3 = \begin{bmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{bmatrix}$

Λ_1 is x-velocity boost

Λ_2 is y-velocity boost

Λ_3 is z-velocity boost

EQUATIONS FOR X-velocity BOOST



Suppose observers (x^μ) and $(x^{\mu'})$ are coincident at the origin and S' moves with constant velocity in x'-direction

$$0 = x' = -ct\sinh\phi + x\cosh\phi$$

$$V = \frac{x}{t} = \frac{ct\sinh\phi}{t\cosh\phi} = c\tanh\phi$$

$$\therefore \frac{V}{c} = \tanh\phi$$

for origin of S' system

$$(\star) \quad \boxed{\begin{aligned} ct' &= ct\cosh\phi - x\sinh\phi \\ x' &= -ct\sinh\phi + x\cosh\phi \end{aligned}}$$

Def^n/ The rapidity of the moving frame is ϕ where $\tanh\phi = \frac{V}{c}$. Also $\beta = V/c$ is a popular notation. We may argue:

$$\gamma = \cosh\phi = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-V^2/c^2}}$$

$$\gamma\beta = \sinh\phi$$

$\star$ yields,

$$\begin{aligned} ct' &= ct\gamma - x\gamma\beta \\ x' &= -ct\gamma\beta + x\gamma \end{aligned} \Rightarrow \begin{aligned} t' &= \gamma(t - \tfrac{x\beta}{c}) \\ x' &= \gamma(x - ct\beta) \end{aligned} \Rightarrow \boxed{\begin{aligned} t' &= \gamma\left(t - \tfrac{xV}{c^2}\right) \\ x' &= \gamma(x - Vt) \end{aligned}}$$

$$\boxed{\Lambda_1 = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\gamma = \frac{1}{\sqrt{1-V^2/c^2}}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = \frac{V}{c}$$

Isometries of Euclidean $\mathbb{R}^3$

Euclidean Distance Function

Def^{n}/ $\mathbb{R}^3$, $d : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ where $\underbrace{d(P,Q) = \sqrt{(P-Q)\cdot(P-Q)}}_{}$ $= \|P-Q\|$ is known as Euclidean 3-space. We say $F : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ when $d(F(P), F(Q)) = d(P,Q)$ $\forall P, Q \in \mathbb{R}^3$

Theorems we can prove (see video 22/70 of my 2021 Diff/Geom. Course, 2-10-21)

1	• Th^{m}/ If $F, G \in \mathrm{Isom}(\mathbb{R}^3)$ then $F \circ G \in \mathrm{Isom}(\mathbb{R}^3)$
2	• Th^{m}/ $T_a : \mathbb{R}^3 \to \mathbb{R}^3$ where $T_a(x) = x + a$ is an isometry for any $a \in \mathbb{R}^3$. That is, every translation on $\mathbb{R}^3$ is an isometry
3	• Th^{m}/ If $R(x) = Ax$ where $A \in O(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^TA = I\}$ then the orthogonal transformation R is an isometry.
4	• Th^{m}/ An isometry fixing 0 is an orthogonal transformation. (minutes $38 \to 53$, this part requires effort)
5	• Th^{m}/ Every isometry of $\mathbb{R}^3$ is the composite of an orthogonal transformation and a translation; $\phi \in \mathrm{Isom}(\mathbb{R}^3)$ then $\exists!$ $a \in \mathbb{R}^3$ and $A \in O(3)$ for which $F(x) = Ax$ and $\phi = T_a \circ F$
6	• Th^{m}/ The set of $\mathrm{Isom}(\mathbb{R}^3)$ is a group under composition. We also call $\phi \in \mathrm{Isom}(\mathbb{R}^3)$ a rigid motion provided $\det A = 1$.

Isometries of Minkowski Space

Def^{n}/ $\mathbb{R}^4$ paired with $d : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ where $d(P,Q) = \sqrt{|g(P-Q, P-Q)|}$ and $g(v,w) = v^T \eta w$ and $\eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ is the Minkowski Metric and we call d the pseudodistance function. We say $F : \mathbb{R}^4 \to \mathbb{R}^4$ is an <u>isometry</u> of Minkowski space when $d(F(P), F(Q)) = d(P,Q)$ for all events $P, Q \in \mathbb{R}^4$

QUESTION : can we adapt results 1, 2, 3, 4, 5 and 6 to the Minkowski context?

spatial translation $\longmapsto$ spacetime translation

orthogonal transformation $O(3)$ $\longmapsto$ Lorentz transformation $O(3,1)$

group of rigid motions and reflections $\longmapsto$ Poincare group

Def^{n}/ $(\mathbb{R}^n, g)$ $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g(x,y) = g(y,x)$ $\forall x,y \in \mathbb{R}^n$, $\underbrace{\det[g] \neq 0}_{\det G \neq 0}$

and $g(v,w) = v^T G w$ then $\{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$ is g-orthogonal if $g(v_i, v_j) = \pm \delta_{ij}$

$\mathbb{R}^4$ $\{e_0, e_1, e_2, e_3\}$ $G = \eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

$g(e_\mu, e_\nu) = \pm \delta_{\mu\nu}$ $\quad g(e_0, e_0) = -1$

$g(e_i, e_i) = 1$

(no sum over i)