Lecture 5: Observers, Euclidean vs. Minkowski Structure

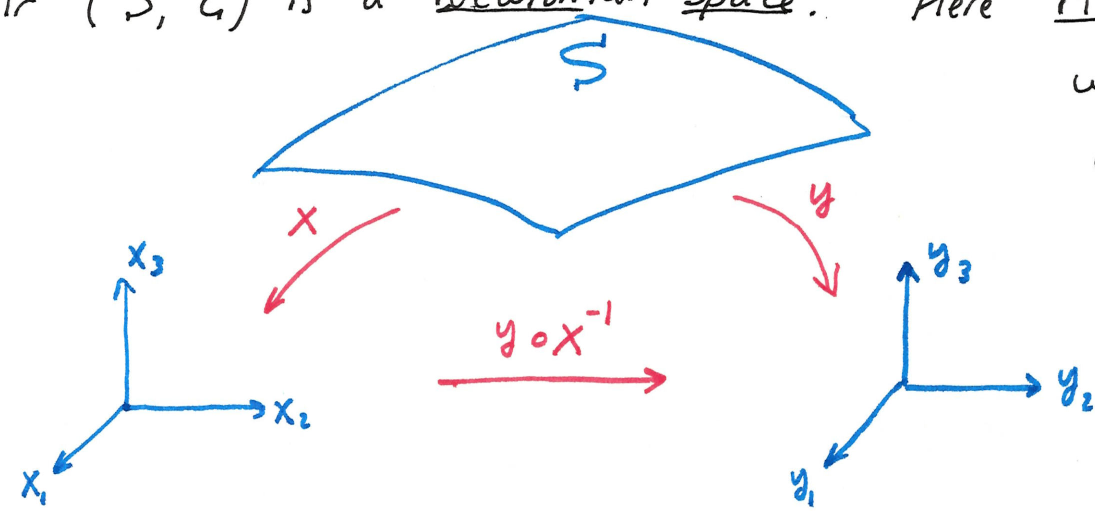
I'll begin by describing a formal version of Newtonian Mechanics I learned from my PhD advisor Dr. R.O. Fulp of NCSU. The key idea is to provide a formalism to handle <u>inertial</u> <u>observers</u>. We first need context,

Def$^{\underline{n}}$/ We say $\mathcal{E}$ is an <u>Euclidean</u> <u>structure</u> on a set S iff $\mathcal{E}$ is a family of bijections from S onto $\mathbb{R}^3$ such that

(1.) if $x, y \in \mathcal{E}$ then $y \circ x^{-1}$ is a rigid motion

(2.) if $x \in \mathcal{E}$ and φ is a rigid motion then $\varphi \circ x \in \mathcal{E}$

The pair $(S, \mathcal{E})$ is a <u>Newtonian</u> <u>Space</u>. Here <u>rigid</u> <u>motion</u> $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ where $\varphi(x) = Rx + a \ \forall x \in \mathbb{R}^3$ and $R \in SO(3) = \{R \mid R^TR = I, \ \det R = 1\}$ and $a \in \mathbb{R}^3$.



Def$^{\underline{n}}$/ An <u>observer</u> is a function from $I \subseteq \mathbb{R}$ into $\mathcal{E}$. Here $x : I \to \mathcal{E}$ is a <u>time</u> varying coordinate system on S. For each $t \in I$ we denote $x(t) = x_t : S \to \mathbb{R}^3$

Given a material particle which moves through $\mathcal{S}$ then if it is at $\gamma(t) \in \mathcal{S}$ at time t then $\gamma : I \longrightarrow \mathcal{S}$ is called the <u>trajectory</u> of the particle.

Defⁿ/ Given an observer $t \mapsto X_t$ and γ the trajectory of a particle then position, velocity and acceleration w.r.t. observer X

1.) position $X_t(\gamma(t))$ at time $t \in I$,
2.) velocity $V_x(t) = \frac{d}{dt}\left[X_t(\gamma(t))\right]$ at time $t \in I$,
3.) acceleration $a_x(t) = \frac{d^2}{dt^2}\left[X_t(\gamma(t))\right]$ at time $t \in I$.

It is interesting to compare position, velocity and acceleration of γ w.r.t. observer X to observer Y. Since $Y_t \circ X_t^{-1}$ is rigid motion for each $t \in I$ there exists $R_t \in SO(3)$ and $P_t \in \mathbb{R}^3$ such that $(Y_t \circ X_t^{-1})(q) = R_t q + P_t$

$$Y_t(\gamma(t)) = R_t X_t(\gamma(t)) + P_t$$

$$Y_t = R_t X_t + P_t$$

$$\boxed{Y = RX + P}$$

$$\frac{dY}{dt} = \frac{d}{dt}(RX + P) = \frac{dR}{dt}X + R\frac{dX}{dt} + \frac{dP}{dt} \qquad \boxed{V_Y = \dot{R}X + RV_X + \dot{P}}$$

$$\frac{d^2Y}{dt^2} = \frac{d}{dt}\left[\dot{R}X + R\dot{X} + \dot{P}\right] = \ddot{R}X + \dot{R}\dot{X} + \dot{R}\dot{X} + R\ddot{X} + \ddot{P}$$

$$\boxed{a_Y = \ddot{R}X + 2\dot{R}V_X + Ra_X + \ddot{P}}$$

The equations relating position, velocity, acceleration between observers x and y where the observers could be non-inertial. Newton's Laws concern <u>inertial</u> <u>observers</u>

Defn/ If $\gamma : I \to \mathcal{S}$ is the trajectory of a particle then we say the particle is in a <u>state of rest relative to an observer $x : I \to \mathcal{E}$</u> iff the mapping from I to $\mathbb{R}^3$ defined by $t \mapsto x_t(\gamma(t))$ is <u>constant</u>. Relative to x, the position vector of γ is not changing. Likewise, the particle experiences <u>uniform</u> <u>rectilinear</u> <u>motion</u> relative to observer $x : I \to \mathcal{E}$ iff $t \mapsto x_t(\gamma(t))$ is a straight line in $\mathbb{R}^3$. Also, the particle <u>experiences</u> <u>a</u> <u>force</u> relative to observer x iff the particle is neither at <u>rest</u> nor in a state of <u>uniform</u> <u>rectilinear</u> <u>motion</u>.

The definition above focuses on properties of a particle.

fixed time-indep. observer

Def^n/ An observer $x : I \to \mathcal{E}$ is called an <u>inertial</u> <u>observer</u> iff $\exists\, x_o \in \mathcal{E}$ and $R \in SO(3)$ and $v, w \in \mathbb{R}^3$ such that $x_t = Rx_o + tv + w \;\; \forall t \in I$. A particle is called a <u>free</u> <u>particle</u> iff it experiences no acceleration relative to an inertial observer.

Th^m/ If $x : I \to \mathcal{E}$ and $y : I \to \mathcal{E}$ are inertial observers then $\exists\, R \in SO(3)$ and $v, w \in \mathbb{R}^3$ such that $y_t = Rx_t + tv + w$ for all $t \in I$. Moreover if a particle experiences no acceleration relative to x then it experiences no acceleration relative to y

NEWTON'S LAWS

- **NEWTON'S FIRST LAW**: every material remains in its state of rest or in its state of uniform rectilinear motion unless compelled by forces acting on it to change its state of motion.

- **NEWTON'S SECOND LAW**: the time rate of change of the product of mass and of the velocity of a material body is proportional to the force acting on the body and takes place in the direction of the straight line along which the force acts.

- **NEWTON'S THIRD LAW**: the forces two bodies exert on each other are always equal and opposite in direction.

The 1st law can be understood to say:

"there exists an inertial observer"

many articles have been written to disect laws 2 & 3.

Remark: the formulas on pg. 2 are very interesting and seldom seen outside a physics course

$$V_y = \dot{R}x + RV_x + \dot{P} \xrightarrow[R=I]{\dot{R}=0} \boxed{V_y = V_x + \dot{P}}$$

$$a_y = \ddot{R}x + 2\dot{R}V_x + Ra_x + \ddot{P} \xrightarrow[R=I]{\dot{R}=0} \boxed{a_y = a_x + \ddot{P}}$$

I teach these in the intro. physics course.

we can dig into the formula for acceleration formula and derive the Coriolis effect and the centripetal or centrifugal effects.

Formal Minkowski Space

Def^{n}/ We say $\mathcal{H}$ is a **Minkowski structure** on a set S iff $\mathcal{H}$ is a family of bijections from S onto $\mathbb{R}^4$ such that

(1.) $x, y \in \mathcal{H}$ then $y \circ x^{-1}$ is the composition of a Lorentz transformation and a spacetime translation

(2.) if $x \in \mathcal{H}$ and φ is composite of Lorentz transformation and spacetime translation then $\varphi \circ x \in \mathcal{H}$

In (2.) we have $\varphi(p) = \Lambda p + q$ where $\Lambda \in SO(3,1)^{\uparrow}$ and $q \in \mathbb{R}^4$. What is an <u>observer</u> in this context? We can't use universal time as before. How should we formalize the construction of an observer on S?