LECTURE 5: INDUCTION

1899 Giuseppe Peano gave five axioms based on successors to describe the natural number system $\mathbb{N}$. In particular,

(i.) 1 is a natural #

(ii.) every natural # has a unique successor

(iii.) no two natural #'s have the same successor

(iv) 1 is not a successor for any natural #

(v.) If a set contains 1 and the successor of every # in the set then the set contains $\mathbb{N}$. } principle of induction.

From the above axioms one may prove (we will not)

Properties of $\mathbb{N}$

1.) for any $x, y \in \mathbb{N}$ the sum $x+y \in \mathbb{N}$ and $x+y = y+x$. } commutative

2.) for any $x, y \in \mathbb{N}$ the product $xy \in \mathbb{N}$ and $xy = yx$

3.) for any $x, y, z \in \mathbb{N}$, $x+(y+z) = (x+y)+z$ } associativity

4.) for any $x, y, z, \in \mathbb{N}$, $x(yz) = (xy)z$

5.) for all $x, y, z \in \mathbb{N}$, $x(y+z) = xy + xz$ } distributive
$(y+z)x = yx + zx$

6.) for all $x, y, z \in \mathbb{N}$, if $x+z = y+z$ then $x = y$. } cancellation
if $xz = yz$ then $x = y$

7.) For all $x, y, z \in \mathbb{N}$, ↘ continued

Properties of $\mathbb{N}$ continued

7.) for all $x, y, z \in \mathbb{N}$,

$x < y$ iff $\exists w \in \mathbb{N}$ s.t. $x + w = y$

$x \leq y$ iff $x < y$ or $x = y$

$x < y$ and $y < z$ implies $x < z$

$x \leq y$ and $y \leq x$ implies $x = y$

if $x < y$ then $x + z < y + z$ and $xz < yz$.

} order Properties.

Remark: you can construct $\mathbb{N}$ from basic set theory and prove all the above results with proper def's laid on foundation of Peano's Postulates. We'll move on now, I mention the above for logical completeness. Our major goal in this Lecture is to understand how to prove statements $\forall n \in \mathbb{N}$. There are three major tools,

- **Induction**: Let $P(n)$ be a proposition which depends on natural number n.
  If $P(1)$ is true and for $n > 1$ we have $P(n)$ true implies $P(n+1)$ true
  then proof by mathematical induction (PMI) provides $P(n)$ true $\forall n \in \mathbb{N}$.

- **Strong Induction**: Let $P(n)$ be a proposition depending on $n \in \mathbb{N}$. Suppose $P(1)$ true and
  if $P(n)$ is true for all $n \leq m$ implies $P(m+1)$ is true then $P(n)$ true $\forall n \in \mathbb{N}$.

- **Well Ordering Principle (WOP)**: every nonempty subset of $\mathbb{N}$ contains a least element.

Example: Claim: $1+2+\cdots+n = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$

Proof: We say P_n is the claim that $1+2+\cdots+n = \frac{1}{2}n(n+1)$

Observe P_1 is true since $1 = \frac{1}{2}(1)(1+1) = \frac{2}{2} = 1$. Inductively assume P_n is true for some $n \in \mathbb{N}$. Consider

$$1+2+\cdots+n+n+1 = \frac{1}{2}n(n+1) + n+1 \quad \text{by induction hypothesis}$$

$$= \frac{1}{2}n(n+1) + \frac{1}{2}(2)(n+1)$$

$$= \frac{1}{2}(n+1)[n+2]$$

$$= \frac{1}{2}(n+1)(n+1+1)$$

Thus P_{n+1} is true and this verifies the inductive step. Therefore, by PMI we find $1+2+\cdots+n = \frac{1}{2}n(n+1)$ $\forall n \in \mathbb{N}$.

Inductive Reasoning: How to Discover Inductive Claim

Example:

$$1 = 1^2$$
$$1+3 = 4 = 2^2$$
$$1+3+5 = 9 = 3^2$$
$$1+3+5+7 = 16 = 4^2$$
$$1+3+5+7+9 = 25 = 5^2$$

Claim: Sum of the 1^{st} n-odd integers is n^2. That is,

$$\sum_{j=1}^{n}(2j-1) = n^2$$

Proof of Claim:

Observe $\sum_{j=1}^{1} 2j-1 = 2-1 = 1^2$ hence the claim holds for $n=1$.

Suppose $\sum_{j=1}^{n} 2j-1 = n^2$ for some $n>1$. Consider the sum of the 1^{st} $(n+1)$-odd integers,

$$\sum_{j=1}^{n+1} 2j-1 = 2(n+1)-1 + \sum_{j=1}^{n}(2j-1) \quad : \text{Def}^{n} \text{ of finite sum.}$$
$$= 2n+1+n^2 \quad : \text{using the induction hypothesis}$$
$$= (n+1)^2$$

Thus the claim$^{\text{true}}$ for n implies the claim true for $n+1$ hence by PMI we conclude $\sum_{j=1}^{n} 2j-1 = n^2 \quad \forall n \in \mathbb{N}$. //

Claim: $\frac{d}{dx}(x^n) = nx^{n-1}$ for all $n \in \mathbb{N}$

Proof: we assume a couple basic results from calculus. Namely that $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$. Let P_n be the statement that $\frac{d}{dx}(x^n) = nx^{n-1}$.

1.) BASE STEP ($n=1$) note $\frac{d}{dx}(x^1) = \frac{dx}{dx} = 1$ thus P_1 true.

2.) INDUCTIVE STEP Let $n > 1$ and assume $\frac{d}{dx}(x^n) = nx^{n-1}$.

Consider, $x^{n+1} = xx^n$ hence,

$$\begin{aligned}
\frac{d}{dx}\left(x^{n+1}\right) &= \frac{d}{dx}\left(xx^n\right) \\
&= \frac{dx}{dx}x^n + x\frac{d}{dx}\left(x^n\right) && \text{: product rule} \\
&= x^n + x \cdot nx^{n-1} && \text{: using induction hypothesis.} \\
&= x^n + nx^n \\
&= (n+1)x^n \\
&= (n+1)x^{n+1-1}
\end{aligned}$$

Thus P_n true implies P_{n+1} true and we find P_n true $\forall n \in \mathbb{N}$ by PMI. //

Example: Show $7^n - 2^n$ is divisible by 5 for all $n \in \mathbb{N}$

P_n

Proof: observe P_1 is true since $7^1 - 2^1 = 5 = 5(1)$ thus $7^1 - 2^1$ is divisible by 5.

Suppose inductively that $7^n - 2^n$ is divisible by 5; $\exists k \in \mathbb{Z}$ for which $7^n - 2^n = 5k$. Consider that $7^n = 2^n + 5k$ and calculate,

$$\begin{aligned} 7^{n+1} - 2^{n+1} &= 7(7^n) - 2(2^n) && : \text{Def}^n \text{ of } a^{n+1} = aa^n \\ &= 7(2^n + 5k) - 2(2^n) && : \text{by induction hypothesis} \\ &= 2^n(7-2) + 7 \cdot 5k \\ &= 2^n(5) + 7(5k) \\ &= 5(2^n + 7k) \end{aligned}$$

and since $2^n + 7k \in \mathbb{Z}$ we find $7^{n+1} - 2^{n+1}$ is divisible by 5 which shows $P_n \Rightarrow P_{n+1}$ and hence P_n is true for all $n \in \mathbb{N}$ by PMI. //

Def^n/ If $a, b \in \mathbb{Z}$ then $a \mid b$ or a <u>divides</u> b means that $\exists k \in \mathbb{Z}$ for which $b = ka$.
Alternatively, can say b is a <u>multiple</u> of a.

Examples → $15 \mid 75$ since $75 = 5(15)$
$6 \nmid 10$ since $\nexists k \in \mathbb{Z}$ s.t. $10 = 6k$.

Recursive Definitions

Some common definitions which use induction,

Def^n of factorial

Def^n/ $0! = 1$ and $n! = n(n-1)!$ $\forall n \in \mathbb{N}$

$0! = 1$
$1! = 1(0)! = 1$
$2! = 2(1)! = 2$
$3! = 3(2)! = 6$
$4! = 4(3)! = 4 \cdot 6 = 24$
$5! = 5(4)! = 5 \cdot 24 = 120$
$6! = 6(5)! = 6 \cdot 120 = 720$
$7! = 7 \cdot 6! = 7(720) = 5040$

Alternatively, you might see someone say,

$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ with $0! \overset{\text{def}^n}{=} 1$.

Th^m/ $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ for all $n \in \mathbb{N}$

Proof: Let P_n be the claim $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

Observe $1! = 1$ which proves the claim for $n = 1$. Suppose P_n true for some $n > 1$ and consider,

$$(n+1)! = (n+1)n! \quad : \text{Def}^n \text{ of factorial}$$
$$= (n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \quad : \text{by induction hypothesis.}$$

Hence P_{n+1} is true and the Th^m follows by PMI. //

Informally we often tell students $\sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$. We can be more careful and define the finite sum recursively

Def^n/ (Finite Sum) Let $a_j \in \mathbb{R}$ for $j \in \mathbb{N}$. We define $\sum_{j=1}^{1} a_j = a_1$ and $\sum_{j=1}^{n} a_j = a_n + \sum_{j=1}^{n-1} a_j$ for $n \geq 2$.

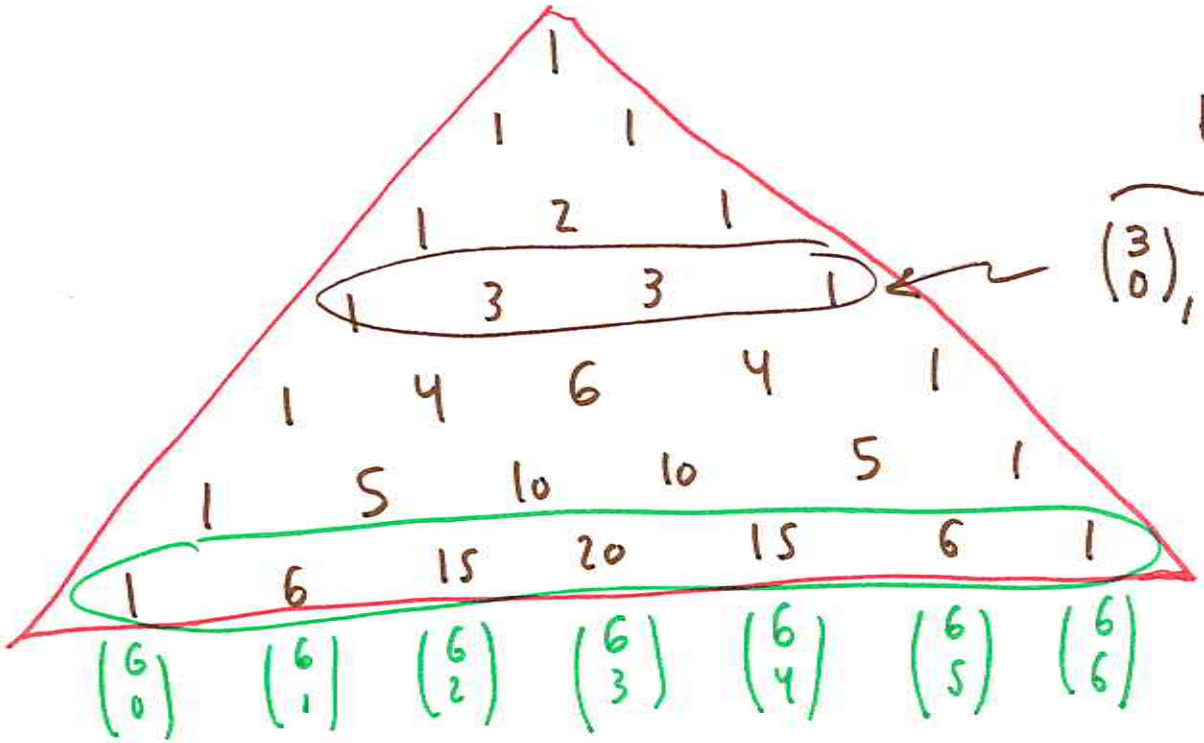
Th^m/ (Linearity of Finite Sums) Let $a_j, b_j, c \in \mathbb{R}$ for $j \in \mathbb{N}$. Then $\sum_{j=1}^{n}(ca_j + b_j) = c\sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j$

Proof: Let P_n be the claim $\sum_{j=1}^{n}(ca_j + b_j) = c\sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j$.

$n=1$/ Notice $\sum_{j=1}^{1}(ca_j + b_j) = ca_1 + b_1 = c\sum_{j=1}^{1} a_j + \sum_{j=1}^{1} b_j$ hence P_1 true.

Suppose $n > 1$ and inductively assume P_n true. Consider,

$$\begin{aligned}
\sum_{j=1}^{n+1}(ca_j + b_j) &= ca_{n+1} + b_{n+1} + \sum_{j=1}^{n}(ca_j + b_j) && \text{: Def}^n \text{ of finite sum.} \\
&= ca_{n+1} + b_{n+1} + c\sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j && \text{: by induction hypothesis.} \\
&= c\left(a_{n+1} + \sum_{j=1}^{n} a_j\right) + b_{n+1} + \sum_{j=1}^{n} b_j && \text{: algebra of } \mathbb{R} \\
&= c\sum_{j=1}^{n+1} a_j + \sum_{j=1}^{n+1} b_j && \therefore P_{n+1} \text{ true and we conclude } P_n \text{ true } \forall n \in \mathbb{N} \text{ by PMI.} //
\end{aligned}$$



binomial coefficients

$\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$

$\{a, b, c\}$

$\{a\}, \{b\}, \{c\}$

$\{a, b\}, \{a, c\}, \{b, c\}$

$\binom{n}{k}$ = # of ways to choose k things from n

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

Lemma: If $\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$ then $\binom{n}{x} + \binom{n}{x-1} = \binom{n+1}{x}$ $\forall x, n \in \mathbb{N}$

Proof: fix $x \in \mathbb{N}$ and let P_n be statement $\binom{n}{x} + \binom{n}{x-1} = \binom{n+1}{x}$.

Observe $(2-x)! = (2-x)(1-x)!$ and $x! = (x-1)!\,x$ for what follows:

$$\binom{1}{x} + \binom{1}{x-1} = \frac{1!}{x!\,(1-x)!} + \frac{1!}{(x-1)!\,(1-(x-1))!} \quad : \text{Def}^{n} \text{ of } \binom{n}{x}$$

$$= \frac{1}{x!\,(1-x)!} + \frac{1}{(x-1)!\,(2-x)!} \quad : 1! = 1$$

$$= \frac{2-x}{x!\,(2-x)!} + \frac{x}{x!\,(2-x)!} \quad : (2-x)! = (2-x)(1-x)! \text{ \& } x! = x(x-1)!$$
$$\frac{1}{(1-x)!} = \frac{2-x}{(2-x)!} \qquad \frac{x}{x!} = \frac{1}{(x-1)!}$$

$$= \frac{2!}{x!\,(2-x)!}$$

$$= \binom{1+1}{x}$$ thus P_1 is true for arbitrary $x \in \mathbb{N}$.

Suppose inductively that $\binom{n}{x} + \binom{n}{x-1} = \binom{n+1}{x}$ for some $n > 1$. Consider,

$$\binom{n+1}{x} + \binom{n+1}{x-1} = \frac{(n+1)!}{x!\,(n+1-x)!} + \frac{(n+1)!}{(x-1)!\,(n+1-(x-1))!} \quad : \text{Def}^{n} \text{ of } \binom{n}{x}$$

$$= \frac{(n+1)!}{x!\,(n+2-x)!}(n+2-x) + \frac{(n+1)!}{x!\,(n+2-x)!}(x) \quad : (n+2-x)! = (n+2-x)(n+1-x)!$$
$$x! = x\,(x-1)!$$

$$= \frac{(n+2)(n+1)!}{x!\,(n+2-x)!}$$

$$= \frac{(n+2)!}{x!\,(n+2-x)!} = \binom{n+1+1}{x}$$ thus P_{n+1} true and the Lemma follows by PMI. //

Th^{m}/ Binomial Th^{m}: (Let $a, b \in \mathbb{R}$) ($\forall n \in \mathbb{N}$)

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \text{where} \quad \underbrace{\binom{n}{k}}_{\text{read "n choose k"}} = \frac{n!}{k!(n-k)!}$$

Proof: Let P_n be the statement $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

Observe $(a+b)^1 = a^1 + b^1 = \binom{1}{0} a^{1-0} b^0 + \binom{1}{1} a^{1-1} b^1$ as $\binom{1}{0} = \frac{1!}{0!\,1!} = 1$ and $\binom{1}{1} = \frac{1!}{1!\,0!} = 1$.

thus P_1 is true and this completes the "base step" for PMI. Next

Suppose $n > 1$ and inductively assume $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Consider,

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n && : \text{Def}^{n} \text{ of } \bar{a}^{n+1} = \bar{a}^n \bar{a},\ \bar{a} = a+b \\
&= (a+b)\left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \right) && : \text{by induction hypothesis} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + b^{n+1} \qquad k = j-1 \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{j=1}^{n} \binom{n}{j-1} a^{n-(j-1)} b^j + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{n+1-k} b^k + b^{n+1}
\end{aligned}$$

Proof: continued

From the Lemma: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ $\forall n \in \mathbb{N}$ and $k \in \mathbb{N}$ we find, continuing our calculation from previous page,

$$(a+b)^{n+1} = a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + b^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k \quad \leftarrow \text{since } \binom{n+1}{0} = \binom{n+1}{n+1} = 1.$$

Thus $P_n \Rightarrow P_{n+1}$ is shown true and we conclude P_n true $\forall n \in \mathbb{N}$ by PMI. //

Remark: typically we don't say "P_n is the statement" when giving induction proofs in mathematics. I'm doing that here to emphasize the logical dependence of the proof on the statement. A typical proof would instead read like:

No mention of P_n

Proof: Show $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ for all $n \in \mathbb{N}$ by induction on n. observe $(a+b)^1 = a+b = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1$ hence $n=1$ holds true. Suppose inductively $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ for some $n > 1$ and consider, ...

Claim: $7^n - 2^n$ is divisible by 5 $\forall n \in \mathbb{N}$

Proof: Notice $7 = 5+2$ thus $7^n \overset{*}{=} \sum_{k=0}^{n} \binom{n}{k} 5^{n-k} 2^k = \sum_{k=0}^{n-1} \binom{n}{k} 5^{n-k} 2^k + 2^n$

thus $7^n - 2^n = \sum_{k=0}^{n-1} \binom{n}{k} 5^{n-k} 2^k = 5\left[\underbrace{\sum_{k=0}^{n-1} \binom{n}{k} 5^{n-k-1} 2^k}_{M}\right]$

Notice $0 \le k \le n-1 \Rightarrow 0 \le n-k-1$ thus $5^{n-k-1} \in \mathbb{N}$ and we find $M \in \mathbb{N}$ which shows $7^n - 2^n = 5M$ where $M \in \mathbb{N}$ $\therefore 5 \mid 7^n - 2^n$ $\forall n \in \mathbb{N}$ //

Remark: the calculation above illustrates that a "$\forall n \in \mathbb{N}$" proof doesn't necessarily require explicit induction-type argument. If we can argue a fact for arbitrary $n \in \mathbb{N}$ then we may justly conclude the fact holds $\forall n \in \mathbb{N}$. Of course, the * step we used the Binomial Th^{m} which is a nontrivial result.

INDUCTION CAN START WHEREVER

Th^m/ $2^n > n^2$ for all $n \in \mathbb{N}$ with $n \geq 5$

← Th^m (1.8) on pg. 11 of Joseph Rotman's "Journey Into Mathematics, An Introduction to Proofs", Dover.

Proof: The base step here is $n = 5$ for $P_n : 2^n > n^2$.

Observe $2^5 = 32 > 25 = 5^2$ thus P_5 is true.

Suppose inductively that $2^n > n^2$ for some $n > 5$.

Let us examine

$$2^{n+1} = 2 \cdot 2^n > 2n^2 \quad \text{(by induction hypothesis)}$$

Notice if $2n^2 > (n+1)^2$ for $n > 5$ then we are done since $2^{n+1} > 2n^2 > (n+1)^2$ would establish P_{n+1} true. Hence consider $n > 5$ and notice

$$2n^2 > (n+1)^2 = n^2 + 2n + 1 \iff n^2 > 2n + 1$$

But, $n^2 = nn > 3n \geq 2n + 1$ as $n > 5 > 3$. Thus $n^2 > 2n+1$ and so $2n^2 > (n+1)^2$ by our arguments above. Therefore we've shown P_n true implies P_{n+1} true and it follows $2^n > n^2$ $\forall n \geq 5, n \in \mathbb{N}$. //

Strong Induction aka Principle of Complete Induction (PCI)

How it works

(1.) Let $S = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$
(2.) Show $1 \in S$; that is show $P(1)$ true.
(3.) For all $n \geq 1$, show $\{1, 2, \ldots, n\} \subseteq S$ implies $n+1 \in S$;
that is show $P(k)$ true for all $k \leq n$ implies $P(n+1)$ true.
(4.) By PCI, $S = \mathbb{N}$; that is, by PCI $P(n)$ is true for all $n \in \mathbb{N}$

How is it different than PMI? The induction hypothesis is assumed for a fixed but arbitrary $n > 1$ in a PMI proof. For PCI we show $P(1)$ true and for $P_1, P_2, \ldots, P_n$ all true to show P_{n+1} also true.

Example: P_n : every natural number $n > 1$ has a prime factor.

Let S be the set of $n \in \mathbb{N}$ such that n has a prime factor and $n > 1$.
Observe $2 \in S$. Let us suppose $\{2, 3, \ldots, n\} \subseteq S$. Consider $n+1$. If $n+1$ is prime then $n+1 \in S$. Else $n+1$ is composite and there exist $a, b \in \mathbb{N}$ such that $a, b \neq 1$ and $n+1 = ab$.
* Clearly $a, b < n$ given $n+1 = ab$ thus $a, b \in S$ and both $a \& b$ have a prime factor which provides $n+1 = ab$ has prime factor $\therefore n+1 \in S$ and we conclude $S = \{2, 3, \ldots\} = \mathbb{N} - \{1\}$ by PCI. //

Is it clear? Suppose $a \geq n$ and $n+1 = ab \geq bn \Rightarrow nb - n \leq 1 \Rightarrow n(b-1) \leq 1$
But, $b \in \mathbb{N}$ and $b \neq 1$ so $b > 1$ and $b - 1 > 0$ so $n(b-1) \leq 1$ is absurd. //

Example: Every positive integer n has factorization $n = 2^k m$ where $k \geq 0$ and m is odd ($m \geq 1$) is the claim P_n.

Proof by complete (strong) induction

BASE STEP: $n = 1 = 2^0(1)$ hence P_1 true.

INDUCTION STEP:

Suppose j has a factorization $2^{k_j} m_j$ for $1 \leq j \leq n$.

Consider $n+1$. If $n+1$ is prime then $n+1 = 2^0(n+1)$ and P_{n+1} true.

If $n+1$ is not prime then $n+1 = ab$ for some $a, b > 1$ with $a, b < n+1$. Notia $a, b < n$ is also clear as $a = n$, $b < n+1$ provides $n+1 = nb \Rightarrow n(1-b) = -1$ for $n, 1-b > 0$ a $\not<$. since $b < n$

Since $a < n$ we have $a = 2^k m$ by the induction hypothesis, and $b = 2^{j'} m'$

Thus $n+1 = 2^k m \, 2^{j'} m' = 2^{k+j'} mm'$ where m, m' are odd and $k + j' \geq 0$. But, the product of odd with odd is odd thus mm' is an odd integer, say $m'' = mm'$ and $n+1 = 2^{k''} m''$ $k + j = k''$

hence P_{n+1} is true and we conclude by PCI that P_n true $\forall n \in \mathbb{N}$. //

Well Ordering Principle (WOP)

Every nonempty subset of $\mathbb{N}$ has a smallest element $\leftarrow$ WOP

Example: Let $n \in \mathbb{N}$ and suppose $n > 1$.

Also assume n is not prime and let S be the set of all factors of n which are not 1. Since n is composite $n = ab$ for some $a, b \in \mathbb{N}$ and $a, b > 1$ hence $a, b \in S \neq \emptyset$. The WOP provides that $\exists\, s_0 \in S$ such that $s_0 \leq s \;\; \forall s \in S$. Suppose s_0 is not prime then $\exists\, a_0, b_0 \in \mathbb{N}$ with $a_0, b_0 > 1$ and $s_0 = a_0 b_0$. Then $a_0, b_0 < s_0$ and yet $n = k s_0 = k a_0 b_0 \Rightarrow a_0$ is factor of $n \Rightarrow a_0 \in S$ which $\rightarrow\leftarrow$ s_0 being the smallest element. Thus s_0 is prime.

- Therefore, any $n > 1$ is either prime or has a prime factor which shows any $n > 1$ has a prime factor.

Th^m (Fundamental Theorem of Arithmetic)

Every $n \in \mathbb{N}$ has a unique upto reordering expansion into a product of r distinct prime powers; $n = (p_1)^{m_1} (p_2)^{m_2} \cdots (p_r)^{m_r}$

ALL HORSES ARE OF SAME COLOR

(Joel E. Cohen, 1961, "On the nature of mathematical proof", Opus. Reprinked in A Random Walk in Science (R.L. Weber ed.), Crane, Russak & Co. 1973.)

P_n: A set of n-horses has horses of one color.

n=1 Certainly a set of one horses has horses with just one color.

Suppose inductively that any set of $n>1$ horses has horses of one color.

Consider a set with $1, 2, 3, \ldots, n, n+1$ horses (number them for discussion)

- Let S be horses $1, 2, 3, \ldots, n$ then by induction hypothesis they have the same color, let's say red for conversation's sake.
- Let T be horses $2, 3, \ldots, n+1$ then by induction hypothesis this set of n-horses also has the same color and it must be red since S & T overlap with horses $2, 3, \ldots, n$ in common.

Hence by PMI we conclude any set of n-horses has horses of just one color. //

WHAT ?