LECTURE 6 : TENSORS

Let V be a vector space over $\mathbb{R}$ and $V^* = \{\alpha : V \to \mathbb{R} \mid \alpha \text{ linear}\}$ its dual space. If $\{e_i\}_{i=1}^n$ is a basis for V then $\{\theta^i\}_{i=1}^n$ is a basis for V^* which is dual to $\{e_i\}_{i=1}^n$ provided $\theta^i(e_j) = \delta_{ij}$

(linear in each V & V^* input)

Def^{n}/

$$\boxed{T : \underbrace{V^* \times V^* \times \cdots \times V^*}_{k\text{-copies}} \times \underbrace{V \times V \times \cdots \times V}_{l\text{-copies}} \longrightarrow \mathbb{R} \quad \Big\} \begin{array}{l} \text{type } (k,l) \\ \text{tensor on } V \end{array}}$$

identify $V = V^{**}$ by the rule $v(\alpha) = \alpha(v)$ $\forall v \in V$ and $\alpha \in V^*$ then we may express T as a sum of tensors of e_i with θ^j,

$$\boxed{T = T^{i_1 \cdots i_k}{}_{j_1 \cdots j_l}\, e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_l}}$$

$T^{i_1 \cdots i_k}{}_{j_1 \cdots j_l}$ are the <u>components</u> of T

Moreover, for $\alpha_1, \ldots, \alpha_k \in V^*$ and $v_1, \ldots, v_l \in V$,

$$T(\alpha_1, \ldots, \alpha_k, v_1, \ldots, v_l) = T^{i_1 \cdots i_k}{}_{j_1 \cdots j_l}\, e_{i_1}(\alpha_1) \cdots e_{i_k}(\alpha_k)\, \theta^{j_1}(v_1) \cdots \theta^{j_l}(v_l)$$

$$= T^{i_1 \cdots i_k}{}_{j_1 \cdots j_l}\, (\alpha_1)_{i_1} \cdots (\alpha_k)_{i_k}\, (v_1)^{j_1} \cdots (v_l)^{j_l}$$

In addition the components of T are given by:

$$T(\theta^{i_1}, \ldots, \theta^{i_k}, e_{j_1}, \ldots, e_{j_l}) = T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}\, \underbrace{e_{a_1}(\theta^{i_1})}_{\delta_{a_1, i_1}} \cdots \underbrace{e_{a_k}(\theta^{i_k})}_{\delta_{a_k, i_k}}\, \underbrace{\theta^{b_1}(e_{j_1})}_{\delta_{b_1, j_1}} \cdots \underbrace{\theta^{b_l}(e_{j_l})}_{\delta_{b_l, j_l}}$$

$$\boxed{T(\theta^{i_1}, \ldots, \theta^{i_k}, e_{j_1}, \ldots, e_{j_l}) = T^{i_1 \cdots i_k}{}_{j_1 \cdots j_l}}$$

<u>Remark</u>: typically $\underbrace{V = T_pM}_{\text{tangent space to } M}$ and $\underbrace{V^* = (T_pM)^*}_{\text{cotangent space to } M}$ and/or we study <u>tensor fields</u> where $p \longmapsto T(p)$ for each p in some subset of M

$\boxed{E1}$ Let $v \in V$ and $\alpha \in V^*$ then $v = v^i e_i$ and $\alpha = \alpha_j \theta^j$ and

$$\alpha(v) = (\alpha_j \theta^j)(v^i e_i) = v^i \alpha_j \theta^j(e_i) = v^i \alpha_j \delta_{ji} = \underline{v^i \alpha_i}.$$

$$v(\alpha) = (v^i e_i)(\alpha_j \theta^j) = v^i \alpha_j\, e_i(\theta^j) = v^i \alpha_j \delta_{ij} = \underline{v^i \alpha_i}.$$

$\left(e_i(\theta^j) \overset{def^n}{=\!=} \theta^j(e_i) = \delta_{ij}\right)$

$\boxed{E2}$ Let $T : V^* \times V^* \times V \times V \longrightarrow \mathbb{R}$ and $\alpha, \beta \in V^*$ and $v, w \in V$ then

$$T(\alpha, \beta, v, w) = \left(T^{ij}{}_{kl}\, e_i \otimes e_j \otimes \theta^k \otimes \theta^l\right)(\alpha, \beta, v, w)$$

$$= \underline{T^{ij}{}_{kl}\, \alpha_i \beta_j v^k w^l}.$$

$$\left.\begin{aligned} e_i(\alpha) &= \alpha_i \\ e_j(\beta) &= \beta_j \\ \theta^k(v) &= v^k \\ \theta^l(w) &= w^l \end{aligned}\right\} \underline{\text{Lemma}}$$

$\boxed{\textbf{Lemma: } \text{for } v \in V \text{ or } \alpha \in V^* \text{ we have } \theta^i(v) = v^i \text{ and } e_i(\alpha) = \alpha_i}$

Proof: Suppose $v = v^j e_j$ then $\theta^i(v) = \theta^i(v^j e_j) = v^j \theta^i(e_j) = v^j \delta_{ij} = v^i$.
Likewise if $\alpha = \alpha_j \theta^j$ then $\alpha(e_i) = (\alpha_j \theta^j)(e_i) = \alpha_j \theta^j(e_i) = \alpha_j \delta_{ji} = \alpha_i$. //

$\boxed{E3}$ Let $g : V \times V \longrightarrow \mathbb{R}$ and $v, w \in V$ then,

$$g(v, w) = (g_{ij}\, \theta^i \otimes \theta^j)(v, w)$$

$$= g_{ij}\, \theta^i(v)\, \theta^j(w)$$

$$= \underline{g_{ij}\, v^i w^j}$$

RAISING & LOWERING INDICES IN MINKOWSKI SPACE

Generally the metric $g_{\mu\nu}$ has inverse $g^{\mu\nu}$ and $g_{\mu\nu} \neq g^{\mu\nu}$. However for Minkowski metric $(\eta_{\mu\nu}) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = (\eta^{\mu\nu})$ so technically $\eta_{\mu\nu} = \eta^{\mu\nu}$.

That said, I'll use both $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ as to foreshadow a best practice when we later have more general metrics.

Def^{n}/ Given α_μ we define $\alpha^\mu = \eta^{\mu\nu}\alpha_\nu$

Given V^μ we define $V_\mu = \eta_{\mu\nu} V^\nu$

Given $F_{\mu\nu}$ we define $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta}$

Details:

$(V^\mu) = (V^0, V^1, V^2, V^3)$

$(V_\mu) = (-V^0, V^1, V^2, V^3)$

$(\alpha_\mu) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$

$(\alpha^\mu) = (-\alpha_0, \alpha_1, \alpha_2, \alpha_3)$

$\begin{pmatrix} \alpha_i = \alpha^i \\ \alpha_0 = -\alpha^0 \end{pmatrix}$

$$F^{0i} = \eta^{0\alpha}\eta^{i\beta} F_{\alpha\beta} = -\delta_{0,\alpha}\,\delta_{i,\beta}\,F_{\alpha,\beta} = -F_{0i}$$

$$F^{ij} = \eta^{i\alpha}\eta^{j\beta} F_{\alpha\beta} = \delta_{i,\alpha}\,\delta_{j,\beta}\,F_{\alpha\beta} = F_{ij}$$

$$F^{00} = \eta^{0\alpha}\eta^{0\beta} F_{\alpha\beta} = (-\delta_{0,\alpha})(-\delta_{0,\beta})\,F_{\alpha\beta} = F_{00}$$

$$F^{i0} = \eta^{i\alpha}\eta^{0\beta} F_{\alpha\beta} = \delta_{i,\alpha}(-\delta_{0,\beta})\,F_{\alpha,\beta} = -F_{i0}$$

$$(F^{\mu\nu}) = \begin{bmatrix} F_{00} & -F_{01} & -F_{02} & -F_{03} \\ -F_{10} & F_{11} & F_{12} & F_{13} \\ -F_{20} & F_{21} & F_{22} & F_{23} \\ -F_{30} & F_{31} & F_{32} & F_{33} \end{bmatrix}$$

Coordinate Change & Scalars...

Vectors transform by $V^{\mu'} = \Lambda^{\mu'}{}_{\nu} V^{\nu}$ (contravariance)

Covectors transform by $\alpha_{\mu'} = \Lambda^{\nu}{}_{\mu'} \alpha_{\nu}$ (covariance)

General tensor transforms by $T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_\ell'} = \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_k'}{}_{\mu_k} \Lambda^{\nu_1}{}_{\nu_1'} \cdots \Lambda^{\nu_\ell}{}_{\nu_\ell'} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$

Let us check on the transformation properties of raised or lowered indices. We need $\Lambda \Lambda^{-1} = I$ and $\eta = \Lambda^T \eta \Lambda$ in terms of index formulas,

$$\Lambda^{\mu}{}_{\nu'} \Lambda^{\nu'}{}_{\rho} = \delta^{\mu}_{\rho} \quad \& \quad \Lambda^{\sigma'}{}_{\lambda} \Lambda^{\lambda}{}_{\tau'} = \delta^{\sigma'}_{\tau'}$$

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_{\rho} \eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\sigma}$$

$$\eta = \Lambda^T \eta \Lambda \hookrightarrow \eta \Lambda^{-1} = \Lambda^T \eta = (\Lambda \eta)^T$$

$X^{\mu'} = \Lambda^{\mu'}{}_{\nu} X^{\nu}$ has <u>inverse</u> transformation $X^{\mu} = \Lambda^{\mu}{}_{\nu'} X^{\nu'}$

Invariance?

$$\begin{aligned}
V^{\mu'} V_{\mu'} &= \eta_{\mu'\alpha'} V^{\mu'} V^{\alpha'} \\
&= \eta_{\mu'\alpha'} (\Lambda^{\mu'}{}_{\nu} V^{\nu})(\Lambda^{\alpha'}{}_{\beta} V^{\beta}) \\
&= \eta_{\mu'\alpha'} \Lambda^{\mu'}{}_{\nu} \Lambda^{\alpha'}{}_{\beta} V^{\nu} V^{\beta} \\
&= (\Lambda^{\mu'}{}_{\nu} \eta_{\mu'\alpha'} \Lambda^{\alpha'}{}_{\beta})(V^{\nu} V^{\beta}) \\
&= \eta_{\nu\beta} V^{\nu} V^{\beta} = V_{\beta} V^{\beta} = V^{\beta} V_{\beta} = V^T \eta V
\end{aligned}$$

Th^{m}/ an expression where <u>covariant</u> (lower) and <u>contravariant</u> (upper) indices are summed over in pairs will give an <u>invariant</u> w.r.t. Lorentz Transformations

COORDINATE CHANGE & GEOMETRIC OBJECTS

- Considering $x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}$ with inverse transformation $x^{\mu} = \Lambda^{\mu}{}_{\nu'} x^{\nu'}$ we have $x^{\mu} = \Lambda^{\mu}{}_{\nu'} \Lambda^{\nu'}{}_{\alpha} x^{\alpha} \Rightarrow \delta^{\mu}_{\alpha} x^{\alpha} = \Lambda^{\mu}{}_{\nu'} \Lambda^{\nu'}{}_{\alpha} x^{\alpha}$

  thus $\boxed{\Lambda^{\mu}{}_{\nu'} \Lambda^{\nu'}{}_{\alpha} = \delta^{\mu}_{\alpha}}$ likewise $\boxed{\Lambda^{\mu'}{}_{\lambda} \Lambda^{\lambda}{}_{\nu'} = \delta^{\mu'}_{\nu'}}$

- If $P = p^{\mu} e_{\mu}$ where $e_0 = (1,0,0,0)$, $e_1 = (0,1,0,0)$, $e_2 = (0,0,1,0)$, $e_3 = (0,0,0,1)$ then $e_{\mu'} = \Lambda^{\mu}{}_{\mu'} e_{\mu}$ is the transformation law for the basis. Notice

$$P = p^{\mu'} e_{\mu'} = (\Lambda^{\mu'}{}_{\nu} p^{\nu})(\Lambda^{\alpha}{}_{\mu'} e_{\alpha}) = \underbrace{\Lambda^{\alpha}{}_{\mu'} \Lambda^{\mu'}{}_{\nu}}_{\delta^{\alpha}_{\nu}} p^{\nu} e_{\alpha} = p^{\alpha} e_{\alpha}$$

the point or <u>event</u> in spacetime exists independent of the particular coordinate systems (x^{μ}) or $(x^{\mu'})$ used to describe P.

- Similarly if $V = V^{\mu} e_{\mu}$ is a vector field on spacetime then $V = V^{\mu} e_{\mu} = V^{\mu'} e_{\mu'}$ since $\boxed{V^{\mu'} = \Lambda^{\mu'}{}_{\nu} V^{\nu} \text{ and } e_{\mu'} = \Lambda^{\nu}{}_{\mu'} e_{\nu}}$

- Likewise, a one-form $\alpha = \alpha_{\mu} \theta^{\mu}$ on spacetime is geometric and $\alpha = \alpha_{\mu'} \theta^{\mu'} = \alpha_{\mu} \theta^{\mu}$ since $\boxed{\begin{array}{c} \alpha_{\mu'} = \Lambda^{\nu}{}_{\mu'} \alpha_{\nu} \\ \hline \theta^{\mu'} = \Lambda^{\mu'}{}_{\nu} \theta^{\nu} \end{array}}$

PARTIAL DERIVATIVE OF TENSOR FIELD

$$X^{\mu'} = \Lambda^{\mu'}{}_{\nu} X^{\nu} \qquad (\text{where } \Lambda \in SO(3,1))$$

$$\frac{\partial X^{\mu'}}{\partial X^{\alpha}} = \underbrace{\frac{\partial}{\partial X^{\alpha}}\left(\Lambda^{\mu'}{}_{\nu}\right)}_{\text{ZERO}} X^{\nu} + \Lambda^{\mu'}{}_{\nu} \underbrace{\frac{\partial X^{\nu}}{\partial X^{\alpha}}}_{\delta^{\nu}_{\alpha}} = \Lambda^{\mu'}{}_{\nu} \delta^{\nu}_{\alpha} = \Lambda^{\mu'}{}_{\alpha}$$

- We find $\boxed{\frac{\partial X^{\mu'}}{\partial X^{\alpha}} = \Lambda^{\mu'}{}_{\alpha}}$ for inertial coordinates in Minkowski Space.

- In curved space, if we study general coordinate transformations of the form $X^{\mu'} = \tilde{\Lambda}^{\mu'}{}_{\nu} X^{\nu}$ then $\frac{\partial X^{\mu'}}{\partial X^{\alpha}} = \frac{\partial \tilde{\Lambda}^{\mu'}{}_{\nu}}{\partial X^{\alpha}} X^{\nu} + \Lambda^{\mu'}{}_{\alpha}$ this spoils the following (outside the context of Minkowski Space)

Th^{m}/ Given type (p,q) tensor $T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$ we find $\partial_\lambda T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$ is a type $(p, q+1)$ tensor. $\left(\partial_\lambda = \frac{\partial}{\partial X^{\lambda}}\right)$

<u>Proof</u>:
$$\partial_{\lambda'} T^{\mu_1' \cdots \mu_p'}{}_{\nu_1' \cdots \nu_q'} = \frac{\partial}{\partial X^{\lambda'}}\left[\Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_p'}{}_{\mu_p} \Lambda^{\nu_1}{}_{\nu_1'} \cdots \Lambda^{\nu_q}{}_{\nu_q'} T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}\right]$$
$$= \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_p'}{}_{\mu_p} \Lambda^{\nu_1}{}_{\nu_1'} \cdots \Lambda^{\nu_q}{}_{\nu_q'} \frac{\partial}{\partial X^{\lambda'}}\left[T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}\right]$$
$$= \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_p'}{}_{\mu_p} \Lambda^{\nu_1}{}_{\nu_1'} \cdots \Lambda^{\nu_q}{}_{\nu_q'} \underbrace{\frac{\partial X^{\lambda}}{\partial X^{\lambda'}}}_{\Lambda^{\lambda}{}_{\lambda'}} \partial_\lambda T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} \;//$$

(<u>Remark</u>: tensor field is attaching tensor at each point. to differentiate we're thinking about varying pts...)

Some Tensor Arithmetic

type (3,2) tensor

Given tensor $T^{\alpha\beta\gamma}{}_{\mu\nu}$ we can define several new tensors by contraction

$T^{\alpha\beta\gamma}{}_{\alpha\nu}$	$T^{\alpha\beta\gamma}{}_{\beta\nu}$	$T^{\alpha\beta\gamma}{}_{\gamma\nu}$
$T^{\alpha\beta\gamma}{}_{\mu\alpha}$	$T^{\alpha\beta\gamma}{}_{\mu\beta}$	$T^{\alpha\beta\gamma}{}_{\mu\gamma}$

type (2,1) tensors

or by a pair of contractions,

$$S^{\beta'} = \Lambda^{\beta'}{}_{\beta} S^{\beta}$$

$T^{\alpha\beta\gamma}{}_{\alpha\beta}$	$T^{\alpha\beta\gamma}{}_{\beta\gamma}$	$S^{\beta} \equiv T^{\alpha\beta\gamma}{}_{\alpha\gamma}$
$T^{\alpha\beta\gamma}{}_{\beta\alpha}$	$T^{\alpha\beta\gamma}{}_{\gamma\beta}$	$T^{\alpha\beta\gamma}{}_{\gamma\alpha}$
γ free	α free	β free

type (1,0) tensors

- These may all be distinct if T has no symmetries, but, when T is symmetric or antisymmetric in particular index pairs then the cases above may overlap.

RAISE INDICES: $T^{\alpha\beta\gamma\sigma\rho} = \eta^{\mu\sigma}\eta^{\nu\rho} T^{\alpha\beta\gamma}{}_{\mu\nu}$

LOWER INDICES: $T_{\sigma\rho\delta\mu\nu} = \eta_{\alpha\sigma}\eta_{\beta\rho}\eta_{\delta\gamma} T^{\alpha\beta\gamma}{}_{\mu\nu}$

Forming a Scalar: $T^{\alpha\beta\gamma\sigma\rho} T_{\alpha\beta\gamma\sigma\rho} = \eta^{\sigma\bar\sigma}\eta^{\rho\bar\rho}\eta_{\alpha\bar\alpha}\eta_{\beta\bar\beta}\eta_{\gamma\bar\gamma} T^{\alpha\beta\gamma}{}_{\bar\sigma\bar\rho} T^{\bar\alpha\bar\beta\bar\gamma}{}_{\sigma\rho}$

Fun with Tensors

Def^n/ $T_{(\mu|\nu|\rho)} = \frac{1}{2}(T_{\mu\nu\rho} + T_{\rho\nu\mu})$

$$T_{\mu(\nu\rho)} = \frac{1}{2}(T_{\mu\nu\rho} + T_{\mu\rho\nu})$$

$$T_{[\mu|\nu|\rho]} = \frac{1}{2}(T_{\mu\nu\rho} - T_{\rho\nu\mu})$$

$$T_{(\alpha\beta\gamma)} = \frac{1}{6}(T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} + T_{\gamma\beta\alpha} + T_{\beta\alpha\gamma} + T_{\alpha\gamma\beta})$$

$$T_{[\alpha\beta\gamma]} = \frac{1}{6}(T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} - T_{\gamma\beta\alpha} - T_{\beta\alpha\gamma} - T_{\alpha\gamma\beta})$$

CAUTION: $T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}$ however, this does not hold for symmetrization / antisym. of three or more indices.

$$T_{\mu\nu\sigma\rho} = T_{(\mu\nu)\sigma\rho} + T_{[\mu\nu]\sigma\rho}$$

$$T_{\mu\nu\sigma\rho} \neq T_{(\mu\nu\sigma)\rho} + T_{[\mu\nu\sigma]\rho}$$ (unless T has a particular structure ...)

Example: suppose $A_{\mu\nu}$ is antisymmetric. "Simplify" the contraction below:

$$T^{\mu\nu}A_{\mu\nu} = (T^{(\mu\nu)} + T^{[\mu\nu]})(A_{\mu\nu})$$

$$= \underbrace{T^{(\mu\nu)}A_{\mu\nu}}_{\text{ZERO}} + T^{[\mu\nu]}A_{\mu\nu} = \frac{1}{2}(T^{\mu\nu}A_{\mu\nu} - T^{\nu\mu}A_{\mu\nu})$$