Lecture 7: The Completeness Axiom for $\mathbb{R}$

Roughly, the completeness of the real numbers describes their adherence to one another, or the idea that as we draw $\longrightarrow \mathbb{R}$ there are no holes. But, this is just intuition,

Def/ Let $A \subseteq \mathbb{R}$. A number M is called an <u>upper bound</u> of A if $x \leq M \ \forall x \in A$. If A has an upper bound then A is said to be <u>bounded above</u>

Likewise,

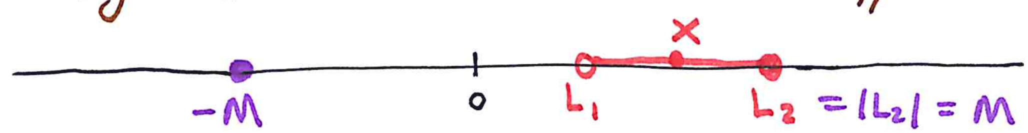
Def/ Let $A \subseteq \mathbb{R}$. A number L is called a <u>lower bound</u> of A if $L \leq x \ \forall x \in A$ and if A has a lower bound then A is said to be <u>bounded below</u>

Finally $A \subseteq \mathbb{R}$ is said to be <u>bounded</u> if A is bounded above and below.

Proposition: $A \subseteq \mathbb{R}$ is bounded iff $\exists M > 0$ such that $|x| \leq M \ \forall x \in A$

Proof: if $A \subseteq \mathbb{R}$ is bounded then $\exists L_1, L_2 \in \mathbb{R}$ for which $L_1 \leq x \leq L_2$ for all $x \in A$. Let $M = \max\{|L_1|, |L_2|\}$ and notice $-|L_1| \leq L_1 \leq x \leq L_2 \leq |L_2|$ provides $-M \leq x \leq M$ as $M \geq |L_1| \Rightarrow -M \leq |L_1|$ and $|L_2| \leq M$ both follow from our construction of M. Thus $-M \leq x \leq M$ then $|x| \leq M$.

Conversely, if $\exists M > 0$ such that $|x| \leq M \ \forall x \in A$ then $-M \leq x \leq M \ \forall x \in A$ and we find A bounded above by M and below by $-M$ hence A is bounded.//



Remark: min and max inequalities of note

$\alpha \leq \max\{\alpha,\beta\}$ and $\beta \leq \max\{\alpha,\beta\}$

$\min\{\alpha,\beta\} \leq \alpha$ and $\min\{\alpha,\beta\} \leq \beta$

$M = \max\{|L_1|, |L_2|\} \geq |L_1|$

$M = \max\{|L_1|, |L_2|\} \geq |L_2|$

Def^{n}/ Let $A \neq \emptyset$, $A \subseteq \mathbb{R}$ and suppose A is bounded above. We call the number α a least upper bound or supremum of A if,

(1.) $x \leq \alpha \quad \forall x \in A$ (α is an upper bound)

(2.) If M is an upper bound of A then $\alpha \leq M$. (α is the least upper bound)

It can be shown such α is unique if it exists, and we denote $\alpha = \sup(A) = \text{lub}(A)$ in the case a least upper bound exists.

Proposition: If $A \subseteq \mathbb{R}$ is nonempty set with suprema α_1 and α_2 then $\alpha_1 = \alpha_2$.

Proof: Since α_1, α_2 are both suprema we have both α_1 & α_2 are upper bounds for A thus $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$ hence $\alpha_1 = \alpha_2$. //

Def^{n}/ Let $\emptyset \neq A \subseteq \mathbb{R}$ and suppose A is bounded below. We call the number β the greatest lower bound (glb) or infimum of A if

(1.) $x \geq \beta \quad \forall x \in A$ (β is a lower bound)

(2.) If L is a lower bound of A then $\beta \geq L$.

When such β exists we write $\inf(A) = \beta = \text{glb}(A)$

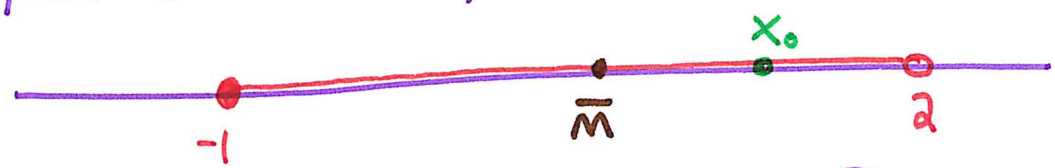
- the proof glb of A is unique if it exists is similar to that I gave for sup?

Examples

1.) $\sup\,[-1,2) = 2$

If $x \in [-1,2)$ then $-1 \le x < 2$ thus $M = 2$ is an upper bound for $[-1,2)$.

Suppose $\overline{M} < 2$ is an upper bound for $[-1,2)$. Construct x_0 as the midpoint of $\overline{M}$ and 2,



$$x_0 = \frac{\overline{M}+2}{2} = 1 + \frac{\overline{M}}{2}$$

observe $\overline{M} \le -1$ provides contradiction as $\overline{M}$ is upper bound on $[-1,2)$.

thus $-1 \le \overline{M} \Rightarrow -\frac{1}{2} \le \frac{\overline{M}}{2} \Rightarrow \frac{1}{2} \le \frac{\overline{M}}{2} + 1 = x_0 \Rightarrow -1 \le x_0$.

Likewise $\overline{M} < 2 \Rightarrow \frac{\overline{M}}{2} < 1 \Rightarrow \frac{\overline{M}}{2} + 1 < 2 \Rightarrow x_0 < 2$

Therefore $x_0 \in [-1,2)$. But we may show $x_0 > \overline{M}$ since

$$\overline{M} = \frac{\overline{M}}{2} + \frac{\overline{M}}{2} < 1 + \frac{\overline{M}}{2} = x_0 \quad \therefore \; \underline{x_0 > \overline{M}}$$

But, $\overline{M}$ is upper bound and $x_0 \in [-1,2)$ $\rightarrow\leftarrow$

Consequently, no such $\overline{M}$ exists and we conclude that $M = 2$ is the <u>least</u> <u>upper</u> <u>bound</u>.

That is, $\underline{\sup\,[-1,2) = 2}$ //

Assumed $\overline{M} < 2$ ... $\rightarrow\leftarrow$

Thus $\overline{M} \not< 2 \Rightarrow \overline{M} \ge 2$.

2.) $\inf[-1,2) = -1$

If $x \in [-1,2)$ then $-1 \leq x < 2$ thus $L = -1$ is a <u>lower</u> <u>bound</u> for $[-1,2)$.

Suppose ℓ is a lower bound for $[-1,2)$

then since $-1 \in [-1,2)$ we find $\ell \leq -1 = L$ thus $L = -1$ is the greatest lower bound and we find $\inf[-1,2) = -1$. //

Remark: when the sup or inf is an element of the set the argument is easier (compare Example 2. vs. Example 1)

3.) $\sup\{2,4,6,8\} = 8$

Clearly if $x \in \{2,4,6,8\}$ then $x \leq 8$ $\therefore$ 8 is upper bound.

If M is any upperbound then $8 \leq M$ thus $M \geq 8 \Rightarrow \sup\{2,4,6,8\} = 8$.

4.) $\sup\{x^2 \mid -2 < x < 1\} = 4$

Let $A = \{x^2 \mid -2 < x < 1\}$

Consider $y = x^2 = |x|^2 < 4$ when $|x| < 2$.

Since $-2 < x < 1 < 2 \Rightarrow |x| < 2$ we have $y = x^2 = |x|^2 < 4 \;\; \forall y \in A$.

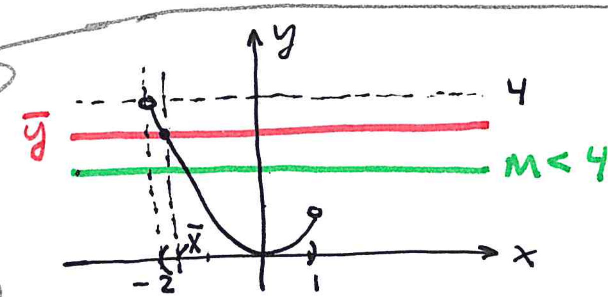
Hence 4 serves as an upperbound for A; or A is bounded above by 4.

Suppose $M < 4$ is an upper bound for A

Consider $\bar{x} = -\sqrt{\frac{M+4}{2}}$ $\left(\bar{y} = \frac{M+4}{2} = \bar{x}^2 \quad \text{want } -2 < \bar{x} < 0 \right)$

Notice, ($M \leq 0$ gives $\rightarrow\leftarrow$ since $0 \in A$)

$$0 < M < 4 \Rightarrow 4 < M+4 < 8$$
$$\Rightarrow 2 < \frac{M+4}{2} < 4$$
$$\Rightarrow \sqrt{2} < \sqrt{\frac{M+4}{2}} < \sqrt{4} = 2$$
$$\Rightarrow -2 < -\sqrt{\frac{M+4}{2}} < -\sqrt{2} < 1$$
$$\Rightarrow \bar{x} = -\sqrt{\frac{M+4}{2}} \in (-2,1) \;\therefore\; \bar{y} = \bar{x}^2 \in A.$$



this picture illustrates my goal,

1.) construct $\bar{y}$ as midpt. between M and 4

2.) show $\bar{y} = \bar{x}^2$ for some $\bar{x} \in (-2,1)$.

3.) prove $\bar{y}$ is above M and yet in A hence $\rightarrow\leftarrow$ M an upper bound.

Yet, $\bar{x}^2 = \frac{M+4}{2} = \frac{M}{2} + 2 = \bar{y}$

and $M < 4 \Rightarrow \frac{M}{2} < 2$ thus $\bar{x}^2 = \frac{M}{2} + 2 > \frac{M}{2} + \frac{M}{2} = M$ hence $\bar{y} = \bar{x}^2 > M$ $\rightarrow\leftarrow$ M an upperbound.

$\therefore \; M \not< 4 \Rightarrow M \geq 4 \Rightarrow \sup(A) = 4.$

Proposition 1.5.1

Let $A \neq \emptyset$ be a subset of $\mathbb{R}$ and let A be bounded above. Then $\alpha = \sup(A)$ iff

(1') $x \leq \alpha$ for all $x \in A$

(2') For any $\varepsilon > 0$, there exists $a \in A$ such that $\alpha - \varepsilon < a$

Proof: ($\Rightarrow$) Assume $\alpha = \sup(A)$ then α is upperbound on A hence 1' holds. Next, let $\varepsilon > 0$ and note $\alpha - \varepsilon < \alpha$ thus $\alpha - \varepsilon$ is not an upper-bound of A. Thus $\exists\, a \in A$ for which $\alpha - \varepsilon < a$ which proves 2' and completes the $\Rightarrow$ portion of the proof.

($\Leftarrow$) Suppose $x \leq \alpha$ for all $x \in A$ and suppose for any $\varepsilon > 0$ there exists $a \in A$ such that $a - \varepsilon < a$. Let $M < \alpha$ be an upper-bound for A. Let $\varepsilon = \alpha - M > 0$ hence $\exists\, a \in A$ such that $a > \alpha - \varepsilon = M$ that is $M < a$ for some $a \in A$. But, M is upper bound $\rightarrow\!\leftarrow$. Thus $M \nless \alpha$ and so $M \geq \alpha$ which shows $\alpha = \sup(A)$ as desired. //

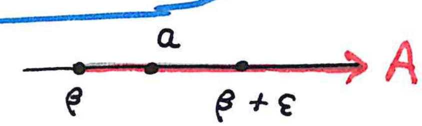
Prop. 1.5.2

Let A be nonempty subset of $\mathbb{R}$ which is bounded below. Then $\beta = \inf(A)$ iff

(1') $\beta \leq x \quad \forall x \in A$

(2') For any $\varepsilon > 0$, there exists $a \in A$ such that $a < \beta + \varepsilon$

Proof: very similar to the one given for suprema. //



Def^{n}/ THE COMPLETENESS AXIOM

Every nonempty subset A of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup(A)$ exists and is a real number.

Example: consider $A = \{ x \mid x$ is decimal representation of $\sqrt{2}$ truncated at n-~~digits~~ decimal places for some $n \in \mathbb{N} \}$

$A = \{1, 1.4, 1.414, \dots\}$ observe $x < \sqrt{2}$ for all $x \in A$. Notice $A \subset \mathbb{Q}$ yet $\sup(A) = \sqrt{2} \notin \mathbb{Q}$. The rational numbers $\mathbb{Q}$ form an ordered field which is not complete.

Th^{m}/ Suppose $A, B \neq \emptyset$ and $A \subseteq B \subseteq \mathbb{R}$. If B is bounded above then $\sup(A) \leq \sup(B)$.

Proof: Suppose B is bounded above by β then $x \leq \beta \ \forall x \in B$. If $x \in A$ then $x \in B$ as $A \subseteq B$. Thus $x \leq \beta \ \forall x \in A$. hence $\underbrace{A \text{ is bounded above}}$.

So … $\sup(A)$ exists by completeness axiom.

Likewise, $\sup(B)$ is upper bound for B it is also an upperbound for A and so by def^{n} (2') for supremum, $\sup(A) \leq \sup(B)$. //

Def^{n}/ Extended Real Number System

$\mathbb{R} \cup \{\infty\} \cup \{-\infty\} = \overline{\mathbb{R}}$

Subject the following conventions, for any $x \in \mathbb{R}$, $-\infty < x < \infty$ and,

(a.) $x + \infty = \infty$, $x + (-\infty) = -\infty$

(b.) $x > 0$ then $x \cdot \infty = \infty$ and $x \cdot (-\infty) = -\infty$

(c.) $x < 0$ then $x \cdot \infty = -\infty$ and $x \cdot (-\infty) = \infty$

(d.) $\infty + \infty = \infty$, $-\infty + (-\infty) = -\infty$, $\infty \cdot \infty = (-\infty)\cdot(-\infty) = \infty$.
$\infty \cdot (-\infty) = (-\infty)\cdot \infty = -\infty$.

If we encounter limits they follow the above rules. However, recall $0 \cdot \infty$, $\infty + (-\infty)$, $(-\infty) + \infty$, are indeterminant forms... we also do not define them here.

Def^{n}/ If $A \neq \emptyset$ is not bounded above in $\mathbb{R}$ we write $\sup(A) = \infty$.
If $A \neq \emptyset$ is not bounded below in $\mathbb{R}$ we write $\inf(A) = -\infty$.

Th^{m}/ Every nonempty subset of $\mathbb{R}$ has a supremum and infimum in the extended real #'s.

Def^{n}/ $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = \infty$.

But, $\emptyset \subseteq A \Rightarrow \sup \emptyset \leq \sup(A) \Rightarrow -\infty \leq \sup(A)$.
$\emptyset \subseteq A \Rightarrow \inf \emptyset \geq \inf(A) \Rightarrow \infty \geq \inf(A)$.