LECTURE 7: 4-VECTORS AND PHYSICS

- Our goal in this lecture is to summarize how Kinematics and Dynamics are modified by Special Relativity. In Newtonian Mechanics we use universal time to frame most physical laws, but, that is no longer an option. Now physical laws must be phrased in a frame-independent, tensorial fashion.
- In the limit of small velocities we should reproduce Newtonian Mechanics.

$$(X^\mu) \equiv (X^0, X^1, X^2, X^3) = (ct, x, y, z) = X$$

$$\bar{A} \cdot \bar{B} = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 = -A^0B^0 + \vec{a} \cdot \vec{b}$$

$$A^2 = \bar{A} \cdot \bar{A} = -(A^0)^2 - a^2 \quad \text{where} \quad a^2 = \vec{a} \cdot \vec{a}$$

$$(A^\mu) = (A^0, \vec{a})$$

A^2:
- $A^2 < 0$ timelike
- $A^2 = 0$ lightlike
- $A^2 > 0$ spacelike

<u>WORLD</u> <u>LINE</u>: $X = X(\tau)$ $\quad$ τ = proper time, measured in comoving ref. frame for particle in question.

- Since $dS^2 = -c\,dt^2 + dx^2 + dy^2 + dz^2$ get $ds^2 = -c^2 d\tau^2$ in <u>rest-frame</u>

$$d\tau^2 = \frac{ds^2}{-c^2} = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2} \Rightarrow \frac{d\tau^2}{dt^2} = 1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]$$

$$\therefore \frac{d\tau}{dt} = \sqrt{1 - u^2/c^2} \quad \text{where} \quad \vec{u} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

Def^n/ $\gamma(u) = \dfrac{1}{\sqrt{1-u^2/c^2}}$ we found $\dfrac{d\tau}{dt} = \dfrac{1}{\gamma(u)}$ or $\dfrac{dt}{d\tau} = \gamma(u)$

4-velocity: $U^\mu = \dfrac{dX^\mu}{d\tau}$ (derivative of world line w.r.t. proper time)

CHAIN RULE: $U^\mu = \dfrac{dX^\mu}{d\tau} = \dfrac{dt}{d\tau}\dfrac{dX^\mu}{dt} = \gamma \dfrac{dX^\mu}{dt}$ * $\vec{u} = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$

$$U^\mu = \gamma(u)\frac{d}{dt}\Big[(ct, x, y, z)\Big] = \gamma(u)\Big(c, \overbrace{\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}}^{\vec{u}}\Big)$$

$$\therefore \boxed{\overline{U} = (\gamma c, \gamma \vec{u}) \text{ where } \gamma = \frac{1}{\sqrt{1-u^2/c^2}}}$$

Remark: $\overline{U}\cdot\overline{U} = -\gamma^2c^2 + \gamma^2\vec{u}\cdot\vec{u} = \gamma^2(u^2-c^2) = \dfrac{u^2-c^2}{1-u^2/c^2} = c^2\left(\dfrac{u^2-c^2}{c^2-u^2}\right)$

thus $\overline{U}\cdot\overline{U} = -c^2$ is an invariant, the 4-velocity has this invariant value no matter what inertial frame we use.

4-ACCELERATION

Def $\frac{n}{-}$ for worldline $X^\mu = X^\mu(\tau)$, $A^\mu = \frac{d^2 X^\mu}{d\tau^2} = \frac{dU^\mu}{d\tau}$

$$\bar{A} = \frac{d\bar{U}}{d\tau} = \gamma \frac{d\bar{U}}{dt} = \gamma \frac{d}{dt}\left(\gamma c, \gamma \vec{u}\right) \quad \text{since } \bar{U} = \gamma(c, \vec{u})$$

$$= \gamma\left(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\vec{u} + \gamma\frac{d\vec{u}}{dt}\right)$$

$$= \underbrace{\gamma\left(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\vec{u} + \gamma\vec{a}\right)}_{\text{where } \gamma = \frac{1}{\sqrt{1-u^2/c^2}}} \quad \text{where } \vec{a} = \frac{d\vec{u}}{dt} = \frac{d^2\vec{x}}{dt}$$

$$\vec{x} = \langle x, y, z \rangle$$

Remark: in the instantaneous rest frame of the particle $u = 0$ and $\gamma = 1$ then we see $\bar{A} = (0, \vec{a})$ where $\vec{a}$ is the acceleration in rest frame

Notice $\bar{A} \cdot \bar{A} = 0$ iff $\vec{a} = 0$.

Th$\frac{m}{-}$ $\bar{U} \cdot \bar{A} = 0$

Proof: $\bar{U} \cdot \bar{U} = -c^2 \Rightarrow \frac{d\bar{U}}{d\tau} \cdot \bar{U} + \bar{U} \cdot \frac{d\bar{U}}{d\tau} = \frac{d}{d\tau}(-c^2) = 0$

$$\therefore \; 2\bar{U} \cdot \frac{d\bar{U}}{d\tau} = 0 \Rightarrow \bar{U} \cdot \bar{A} = 0 \,.//$$

4-ACCELERATION CONTINUED

Th^m/ If $\bar{A} = (A^0, \vec{A})$ then $A^0 = \dfrac{\vec{u}\cdot\vec{A}}{c}$ and $\vec{A} = \gamma^2\vec{a} + \gamma^4\left(\dfrac{\vec{u}\cdot\vec{a}}{c^2}\right)\vec{u}$ where $\vec{u} = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right\rangle$ and $\vec{a} = \dfrac{d\vec{u}}{dt} = \langle \ddot{x}, \ddot{y}, \ddot{z}\rangle$ and $\gamma = \dfrac{1}{\sqrt{1-u^2/c^2}}$

Proof: from ③ we have $\bar{A} = \left(c\gamma\dfrac{d\gamma}{dt}, \gamma\dfrac{d\gamma}{dt}\vec{u} + \gamma^2\vec{a}\right)$ for $\gamma = \dfrac{1}{\sqrt{1-u^2/c^2}}$

Since $\bar{U}\cdot\bar{A} = -U^0A^0 + U^1A^1 + U^2A^2 + U^3A^3$, $\bar{U} = (c\gamma, \gamma\vec{u})$

$$\Rightarrow \quad 0 = -c\gamma A^0 + \gamma\vec{u}\cdot\vec{A}$$

$$\therefore \quad \boxed{A^0 = \frac{\vec{u}\cdot\vec{A}}{c}}$$

Observe
$$\gamma\frac{d\gamma}{dt} = \frac{1}{2}\frac{d}{dt}(\gamma^2) = \frac{1}{2}\frac{d}{dt}\left[\frac{1}{1-u^2/c^2}\right]$$
$$= \frac{1}{2}\frac{-1}{(1-u^2/c^2)^2}\frac{d}{dt}\left[1 - \frac{\vec{u}\cdot\vec{u}}{c^2}\right]$$
$$= \frac{-1}{2}\gamma^4\left(\frac{-1}{c^2}\left(\frac{d\vec{u}}{dt}\cdot\vec{u} + \vec{u}\cdot\frac{d\vec{u}}{dt}\right)\right)$$
$$= \frac{1}{c^2}\gamma^4\vec{u}\cdot\vec{a}$$

$$\therefore \quad \underline{\vec{A} = \gamma^2\vec{a} + \frac{1}{c^2}\gamma^4(\vec{u}\cdot\vec{a})\vec{u}} \quad //$$

<u>Example: velocity transformation for 3-velocity</u>

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$

$$\bar{U} = (\gamma c, \gamma \vec{u}) = (\gamma c, \gamma u_1, \gamma u_2, \gamma u_3) = (U^0, U^1, U^2, U^3)$$

$$u_1 = c\frac{U_1}{U_0} \quad \Big\| \quad u_2 = c\frac{U_2}{U_0} \quad \Big\| \quad u_3 = c\frac{U_3}{U_0} \quad *$$

Lorentz transformation to usual x-boost by V, $\gamma(v) = \dfrac{1}{\sqrt{1-V^2/c^2}}$

$$(U^0)' = \gamma(v)\left(U^0 - VU^1/c\right)$$

$$(U^1)' = \gamma(v)\left(U^1 - VU^0/c\right)$$

$$(U^2)' = U^2$$

$$(U^3)' = U^3$$

Following $*$ for boosted frame,

$$u_1' = c\frac{(U^1)'}{(U^0)'} = c\left(\frac{U^1 - VU^0/c}{U^0 - VU^1/c}\right)\frac{\gamma(v)}{\gamma(v)}$$

$$= c\left(\frac{\gamma u_1 - V\gamma c/c}{\gamma c - Vu_1\gamma/c}\right) \leftarrow \gamma = \frac{1}{\sqrt{1-u^2/c^2}} \text{ all cancel out.}$$

$$\therefore \quad \boxed{u_1' = \frac{u_1 - V}{1 - Vu_1/c^2}}$$

Likewise, $u_2' = \dfrac{c(U_2^2)'}{(U^0)'} = \dfrac{\gamma u_2 c}{\gamma(v)(\gamma c - V\gamma u_1/c)} = \dfrac{u_2}{\gamma(v)[1 - Vu_1/c^2]}$

$\therefore \quad \boxed{u_2' = \dfrac{u_2\sqrt{1-V^2/c^2}}{1 - Vu_1/c^2}}$ and similarly $\boxed{u_3' = \dfrac{u_3\sqrt{1-V^2/c^2}}{1 - Vu_1/c^2}}$

4- MOMENTUM

Def^n/ $\bar{P} = m\bar{U} = m\gamma(c, \vec{u}) = (m\gamma c, m\gamma\vec{u}) = (P^0, \vec{P})$
and $\gamma = \dfrac{1}{\sqrt{1-u^2/c^2}}$. Here $\bar{P}$ is the 4-momentum and $\vec{P}$ is the <u>relativistic momentum</u>

<u>Remark</u>: Conservation of 4-momentum

① $\sum_{\text{before}} m\gamma c = \sum_{\text{after}} m\gamma c$

② $\sum_{\text{before}} \vec{P} = \sum_{\text{after}} \vec{P}$

- For $u \approx 0$ we have $\gamma \approx 1$
  then ① is conservation of mass
- For general u we can show $m\gamma(u) = m + \frac{1}{c^2}\left(\frac{1}{2}m\gamma u^2\right) + \cdots$

$$\begin{aligned} m\gamma c^2 &= mc^2\left(1 - u^2/c^2\right)^{-1/2} \\ &= mc^2\left(1 - \frac{1}{2}\left(\frac{-u^2}{c^2}\right) + \cdots\right) \\ &= \underbrace{mc^2}_{\text{REST ENERGY}} + \underbrace{\frac{1}{2}mu^2}_{\text{Kinetic Energy in Newtonian Mechanics}} + \cdots \end{aligned}$$

Def^n/ $E = m\gamma c^2$ is the relativistic energy for massive particle.

① is conservation of relativistic energy.

Def^n/ a particle with relativistic energy E and momentum $\vec{P}$ has 4-momentum $\bar{P} = (E/c, \vec{P})$

(still makes sense when $m = 0$, e.g. light where $E = hf$)

Th^{m}/ $E^2 = c^2p^2 + m^2c^4$ where $E = m\gamma c^2$ and $p^2 = \vec{p}\cdot\vec{p}$ where m is the rest mass and $\gamma = (1 - u^2/c^2)^{-1/2}$

Proof: $\bar{P} = (m\gamma c, \vec{p})$ is the 4-momentum. Notice $\bar{P}\cdot\bar{P}$ is a Lorentz invariant and $\bar{P}\cdot\bar{P} = -m^2\gamma^2c^2 + \vec{p}\cdot\vec{p}$. However, in the rest frame of m we have $\vec{p} = 0$ and $\gamma = 1$ thus $\bar{P}\cdot\bar{P} = -m^2c^2$. Observe that $m\gamma c = E/c$ thus $-(E/c)^2 + p^2 = -m^2c^2$ $\therefore$ $\underline{E^2 = c^2p^2 + m^2c^4}$ //

Remark: for massless particle, $\bar{P} = (E/c, \vec{p})$ have $\bar{P}\cdot\bar{P} = 0$ and $E = pc$.

Th^{m}/ Elastic collision of two particles has Lorentz invariant $\bar{P}_1\cdot\bar{P}_2$

Proof: $\underbrace{\bar{P}_1 + \bar{P}_2}_{\text{before}} = \underbrace{\bar{P}_1' + \bar{P}_2'}_{\text{after}}$ $\Rightarrow$ $(\bar{P}_1 + \bar{P}_2)\cdot(\bar{P}_1 + \bar{P}_2) = (\bar{P}_1' + \bar{P}_2')\cdot(\bar{P}_1' + \bar{P}_2')$

$$\Rightarrow \cancel{\bar{P}_1\cdot\bar{P}_1} + \underline{2\bar{P}_1\cdot\bar{P}_2} + \cancel{\bar{P}_2\cdot\bar{P}_2} = \cancel{\bar{P}_1'\cdot\bar{P}_1'} + \underline{2\bar{P}_1'\cdot\bar{P}_2'} + \cancel{\bar{P}_2'\cdot\bar{P}_2'}$$

$$\therefore \bar{P}_1\cdot\bar{P}_2 = \bar{P}_1'\cdot\bar{P}_2'$$

4-Force and Newton's Law in Special Relativity

Def^{n}/ $k_i = \frac{dP_i}{dt} = \frac{d}{dt}(m\gamma u_i)$: relativistic 3-force

$$K^\alpha = \frac{dP^\alpha}{d\tau} = \frac{d(mV^\alpha)}{d\tau} = \gamma(u)\left(\frac{1}{c}\frac{dE}{dt}, k_i\right)$$: relativistic 4-force

Robert Resnick
Introduction to Special Relativity
pg. 124

Let $\vec{k} = \vec{F}$ be the force on mass m_o with velocity $\vec{u}$
then Newton's Law is given by $\frac{d\vec{P}}{dt} = \frac{d}{dt}(m_o\gamma\vec{u})$

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(m_o\gamma\vec{u}) = m_o\frac{d\gamma}{dt}\vec{u} + m_o\gamma\frac{d\vec{u}}{dt} = \vec{F}$$

But, $E = m_o\gamma c^2$ thus $\frac{dE}{dt} = m_o c^2\frac{d\gamma}{dt} = \frac{dK}{dt} = \frac{d}{dt}\int \vec{F}\cdot d\vec{l} = \vec{F}\cdot\frac{d\vec{l}}{dt}$

work energy Th$^{\underline{m}}$

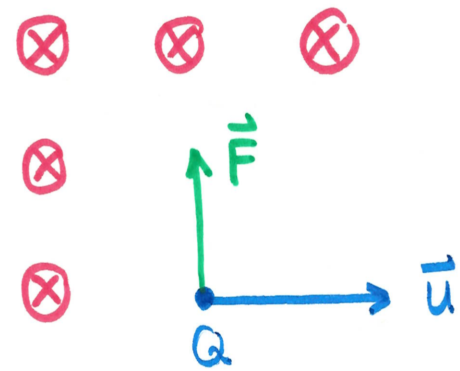
thus $m_o c^2\frac{d\gamma}{dt} = \vec{F}\cdot\vec{u}$

$$\therefore \frac{d\vec{P}}{dt} = \vec{F} = \frac{(\vec{F}\cdot\vec{u})\vec{u}}{c^2} + m_o\gamma\frac{d\vec{u}}{dt}$$

$$\boxed{\vec{a} = \frac{\vec{F}}{m_o\gamma} - \frac{(\vec{F}\cdot\vec{u})\vec{u}}{m_o\gamma c^2}}$$

Recall: $\frac{dt}{d\tau} = \gamma(u)$
then we find that
$$\frac{d\vec{P}}{dt} = \frac{d\vec{P}}{d\tau}\frac{d\tau}{dt} = \frac{1}{\gamma(u)}\frac{d\vec{P}}{d\tau}$$
$$\therefore \frac{d\vec{P}}{d\tau} = \gamma(u)\frac{d\vec{P}}{dt} = \gamma(u)\vec{F}$$

Example: circular motion of charge in B-field



← uniform B-field into page

$$\vec{F} = Q\vec{u}\times\vec{B}$$

(Lorentz' force law)
(RIGHT HAND RULE)

centripetal acceleration formula.

$$F = QuB = m_0\gamma a = m_0\gamma\frac{u^2}{R}$$

(From pg. 8 the $\vec{F}\cdot\vec{u}$ term vanishes since $\vec{F}\perp\vec{u}$ here)

$$R = \frac{m_0\gamma u^2}{QuB}$$

$$\boxed{R = \frac{m_0 u}{QB\sqrt{1-u^2/c^2}}}$$

See pg. 128 of Resnick for Bucherer's Results which experimentally confirm this formula.

Remark: you can solve $a\gamma = \frac{QB}{m_0}u$ directly if you like differential equations.

$$Q\vec{u}\times\vec{B} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left[m_0\frac{\vec{u}}{\sqrt{1-u^2/c^2}}\right]$$

$$= \underbrace{m_0\frac{d\gamma}{dt}\vec{u}}_{\text{must be zero.}} + m_0\gamma\frac{d\vec{u}}{dt}$$

$$U = \gamma u$$

$$QuB = m_0\gamma\frac{du}{dt} = m_0\frac{d}{dt}[\gamma u]$$

multiply by γ,

$$Q\gamma u = m_0\gamma\frac{dU}{dt} = m_0\frac{dU}{d\tau}$$

Converting t - DEq^{ns} to τ - DEq^{ns} is key

$$Q\,\vec{u}\times\vec{B} = \frac{d\vec{P}}{dt} = \frac{d}{dt}\left[m_o\gamma\vec{u}\right] \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$

$$= m_o\frac{d\gamma}{dt}\vec{u} + m_o\gamma\frac{d\vec{u}}{dt} \qquad \Rightarrow \frac{d\gamma}{dt} = 0$$

$$= m_o\frac{d}{dt}\left[\gamma\vec{u}\right]$$

Multiply by γ,

$$Q\,(\gamma\vec{u})\times\vec{B} = m_o\gamma\frac{d}{dt}\left[\gamma\vec{u}\right]$$

Since $dt = \gamma\, d\tau$

$$\gamma\frac{d}{dt} = \frac{d}{d\tau}$$

$$Q\,\vec{U}\times\vec{B} = m_o\frac{d\vec{U}}{d\tau}$$

$$\vec{U} = \frac{d\vec{X}}{d\tau} \quad \text{so} \quad \frac{d\vec{U}}{d\tau} = \frac{d^2\vec{X}}{d\tau^2}$$

Choose coordinates such that $\vec{B} = \langle 0, 0, B\rangle$ we face:

$$\langle \dot{x}, \dot{y}, \dot{z}\rangle \times \langle 0, 0, B\rangle = \frac{m_o}{Q}\langle \ddot{x}, \ddot{y}, \ddot{z}\rangle$$

$$\boxed{\langle B\dot{y}, -B\dot{x}, 0\rangle = \frac{m_o}{Q}\langle \ddot{x}, \ddot{y}, \ddot{z}\rangle} \quad (*)$$

Continuing from * on 10,

$$\ddot{x} = \frac{QB}{m_0}\dot{y} = \alpha \dot{y} \qquad \left(\alpha \equiv \frac{QB}{m_0}\right)$$

$$\ddot{y} = \frac{-QB}{m_0}\dot{x} = -\alpha \dot{x} \rightarrow \dddot{y} = -\alpha \ddot{x} = -\alpha(\alpha \dot{y})$$

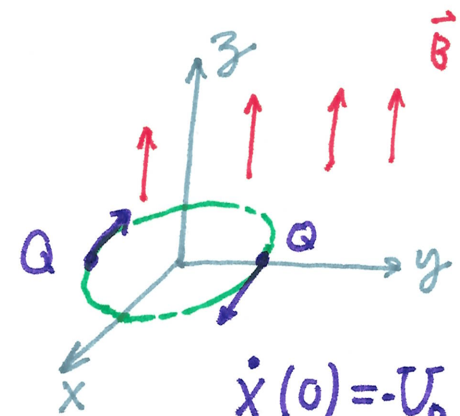
Let $V_y = \dot{y}$ then $\ddot{V}_y + \alpha^2 V_y = 0$

$$\therefore V_y(\tau) = A\sin(\alpha\tau + \phi)$$

$$\ddot{x} = \alpha\dot{y} = \alpha A \sin(\alpha\tau + \phi)$$

$$\dot{x} = -A\cos(\alpha\tau + \phi) + C_1$$

$$x = -\frac{A}{\alpha}\sin(\alpha\tau + \phi) + C_1\tau + C_2$$



$\dot{x}(0) = -U_0$

$\dot{y}(0) = 0$

$$V_y = \dot{y} = A\sin(\alpha\tau + \phi) \Rightarrow y = \frac{-A}{\alpha}\cos(\alpha\tau + \phi) + C_3$$

Suppose we have initial conditions giving $C_1 = C_2 = C_3 = 0$. Then

$$\underbrace{x^2 + y^2 = \frac{A^2}{\alpha^2}(\sin^2\theta + \cos^2\theta) = \left(\frac{A}{\alpha}\right)^2 = R^2}_{\text{CIRCULAR MOTION}} \Rightarrow \boxed{R = \frac{|A|}{\alpha}}$$

Given $\dot{x}(0) = -U_0 = -A\cos\phi = -A$ (set $\phi = 0$)

$$R = \frac{U_0}{\alpha} = \frac{U_0 m_0}{QB} \qquad U_0 = \gamma u_0$$

$$\boxed{R = \frac{m_0 u_0}{QB\sqrt{1 - u_0^2/c^2}}}$$