LECTURE 8 : MAXWELL'S EQUATIONS & ENERGY / MOMENTUM (§1.8 & 1.9 Carroll)

Maxwell added a term to the Eq's governing electric and magnetic fields which ultimately implied understanding light as an E&M wave. It seems Carrol is using units where μ_0 and ϵ_0 are omitted

permeability of the vacuum (μ_0) — permitivity of the vacuum (ϵ_0)

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$\nabla \times \vec{B} - \partial_t \vec{E} = \vec{J}$$
$$\nabla \cdot \vec{E} = \rho$$
$$\nabla \times \vec{E} + \partial_t \vec{B} = 0$$
$$\nabla \cdot \vec{B} = 0$$

$$\tilde{\epsilon}^{ijk} \partial_j B_k - \partial_0 E^i = J^i$$
$$\partial_i E^i = J^0$$
$$\tilde{\epsilon}^{ijk} \partial_j E_k + \partial_0 B^i = 0$$
$$\partial_i B^i = 0$$

$$F^{0i} = E^i = -F^{i0}$$
$$F^{ij} = \tilde{\epsilon}^{ijk} B_k$$

Define the field strength tensor $F^{\mu\nu}$ and

$$J^\mu = (\rho, J^1, J^2, J^3)$$

is the current 4-vector

$\vec{E}$ = electric field

$\vec{B}$ = magnetic field

$\vec{J}$ = current density

$\rho = \frac{dQ}{dV}$ charge density

$$\boxed{\begin{aligned} \partial_j F^{ij} - \partial_0 F^{0i} &= J^i \\ \partial_i F^{0i} &= J^0 \end{aligned}}$$

(Ampere's Law with Maxwell's Correction)

(Gauss' Law)

$$\boxed{\partial_\mu F^{\nu\mu} = J^\nu}$$

$$\boxed{\begin{aligned} G^{0i} &= B^i = -G^{i0} \\ G^{ij} &= \tilde{\epsilon}^{ijk} E^k \\ \partial_\mu G^{\nu\mu} &= 0 \end{aligned}}$$

then see for the other two Maxwell Eq's

(not in Carroll, but the Dual Tensor $G^{\mu\nu}$ is natural to consider)

$$[F^{\mu\nu}] = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$

$$[F_{\mu\nu}] = [\eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}]$$

$$= \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$

$$F_{0i} = -E^i = -F_{i0}$$

$$F_{ij} = \tilde{\epsilon}^{ijk}B^k$$

$$\partial_{[\mu}F_{\nu\lambda]} = 0$$

$$\underbrace{\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0}$$

this gives the homogeneous Maxwell Eq^{n}'s. Not entirely obvious... let's break it down

Consider $\mu = 0, \nu = 0, \lambda = i$

$$\underbrace{\partial_0 F_{0i} + \partial_0 F_{i0} + \partial_i F_{00} = 0}_{\text{tautology. } F_{0i} = -F_{i0}}$$

Consider $\mu = 0, \nu = i, \lambda = j$

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0$$

$$\partial_0 \tilde{\epsilon}^{ijk}B^k + \partial_i(+E^j) + \partial_j(-E^i) = 0$$

$\underline{i=1, j=2}$ $\quad \partial_0 B^3 + \partial_1 E^2 - \partial_2 E^1 = 0$

$$(\nabla \times \vec{E})^3 + \partial_0 B^3 = 0$$

Recall that $\epsilon\epsilon = \delta\delta - \delta\delta$,

$$\epsilon^{ijk}\epsilon^{kab} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$$

$$\partial_0(\epsilon^{abi}\epsilon^{ijk}B^k) + \partial_i \epsilon^{abi}E^j - \partial_j \epsilon^{abi}E^i = 0$$

$$\partial_0((\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj})B^k) + \epsilon^{abi}\partial_i E^j - \epsilon^{abi}\partial_j E^i = 0$$

anyway... there are better methods...

Energy and Momentum : from particles to stress-energy tensor

We've done some of this in LECTURE 7, but I summarize §1.9 of Carroll here:

$x^\mu(\tau)$ world line of massive particle parametrized by proper time τ

$U^\mu = \frac{dx^\mu}{d\tau}$: 4-velocity $\qquad U^\mu U_\mu = -1$ (Carroll sets $c=1$)

$p^\mu = m\,U^\mu$: 4-momentum $\qquad p^\mu p_\mu = -m^2$ & $E = \sqrt{m^2 + p^2}$

$$f^\mu = m\frac{d^2}{d\tau^2}(x^\mu) = \frac{dp^\mu}{d\tau}$$

$\vec{f} = q(\vec{E} + \vec{v}\times\vec{B})$ generalizes to $f^\mu = -q\,U^\lambda F_\lambda{}^\mu$ (tensorial eq^{n})

Extended systems made of many particles can be described as a <u>fluid</u> which is characterized by <u>density</u>, <u>pressure</u>, <u>entropy</u>, <u>viscosity</u> etc...

$T^{\mu\nu}$ denotes the <u>energy-momentum tensor</u>
it is a symmetric $(2,0)$-tensor.
"the flux of 4-momentum p^μ accross surface of constant x^ν"

all comments for rest frame of a fluid

T^{00} is flux of p^0 in the x^0 direction, the rest frame energy density

$T^{0i} = T^{i0}$ is the momentum density

$T^{ij} = T^{ji}$ gives forces between neighboring fluid elements, that is <u>stress</u>

T^{ii} gives pressure

T^{ij} $(i \neq j)$ gives shearing forces

DUST : Collection of particles at rest w.r.t. each other

$U^\mu(x)$ should be constant as x ranges over the dust.

$N^\mu = n\,U^\mu$: "number-flux four-vector"
(n = # density of particles as measured in their rest frame.)

Since $U^\mu = (\gamma(u)c, \gamma(u)\vec{u})$ in the rest frame $\vec{u} = 0$, $\gamma = 1$ and $c = 1$ (Carroll)
then $N^0 = n\,U^0 = n$ in the rest-frame.

N^0 measured in another frame gives # density in that frame
N^i measures flux of particles in x^i-direction

Assume all the particles have mass m then

$\rho = mn$ = rest frame energy density

But, $p^\mu = (m,0,0,0)$ and $N^\mu = (n,0,0,0)$ are vectors in the rest frame
for which $p^\mu \otimes N^\nu$ has $(0,0)$-component (mn) of energy density... so...

$$\boxed{T^{\mu\nu}_{\text{dust}} = p^\mu N^\nu = mn\,U^\mu U^\nu = \rho\,U^\mu U^\nu}$$ ← stress-energy tensor for dust.

PERFECT FLUID

- constant rest frame energy density ρ
- isotropic rest frame pressure P

$$\boxed{T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}}$$

← Stress-Energy Tensor of perfect fluid in its rest frame

What is the general form of $T^{\mu\nu}$ though?

since $T^{\mu\nu}_{\text{Dust}} = \rho\, U^\mu U^\nu \longrightarrow T^{\mu\nu}_{\substack{\text{naive} \\ \text{perfect fluid}}} = (\rho + P) U^\mu U^\nu = \begin{bmatrix} \rho+P & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$

we need to obtain the boxed eqn for the right formulation ...

$$\boxed{T^{\mu\nu} = (\rho + P) U^\mu U^\nu + P\eta^{\mu\nu}}$$

← tensorial and reduces to $T^{\mu\nu}$ in rest frame for perfect fluid as we want.

Defn/ relating pressure P to energy density ρ is known as an <u>equation</u> of <u>state</u>; $P = P(\rho)$

DUST : $P = 0$

ISOTROPIC GAS OF PHOTONS : $P = \frac{1}{3}\rho$

VACUUM ENERGY : $P_{vac} = -\rho_{vac}$

CONSERVATION LAWS

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$ ← conservation of Stress-Energy tensor as 4-dim'l divergence. We should unpack this Physical Equation.

Basically, $\nu = 0$ gives conservation of energy whereas $\nu = 1, 2, 3$ describes conservation of momentum.

PERFECT FLUID : $T^{\mu\nu} = (p+\rho) U^\mu U^\nu + p\,\eta^{\mu\nu}$

$$\partial_\mu T^{\mu\nu} = (p+\rho)\left[(\partial_\mu U^\mu) U^\nu + U^\mu \partial_\mu U^\nu\right] + p\,\partial_\mu \underbrace{(\eta^{\mu\nu})}_{0} + \partial_\mu (p+\rho) U^\mu U^\nu + (\partial_\mu p)\eta^{\mu\nu}$$

0 in flat spacetime, η is constant matrix

(oops, I forgot p, ρ aren't constants) WAIT... I thought they were constants... oh carry on...

Notice $U^\mu U_\mu = -1 \Rightarrow \partial_\nu (U^\mu U_\mu) = 0$

or $(\partial_\nu U^\mu) U_\mu + U^\mu (\partial_\nu U_\mu) = 0 \Rightarrow U_\nu \partial_\mu U^\nu = 0$

$$U_\nu \partial_\mu T^{\mu\nu} = (p+\rho) U_\nu U^\nu \partial_\mu U^\mu + 0 + \partial_\mu (p+\rho) U^\mu U_\nu U^\nu + (\partial_\mu p) U_\nu \eta^{\mu\nu}$$

$$= -(p+\rho)\partial_\mu U^\mu - \partial_\mu (p+\rho) U^\mu + (\partial_\mu p) U^\mu$$

Some simplifications

$$= -\partial_\mu (\rho U^\mu) - p\,\partial_\mu U^\mu = 0 \quad *$$

Ok, now take non-relativistic limit where $U^\mu = (1, v^i)$, $|v^i| \ll 1$, $p \ll \rho$

$*$: $$\boxed{\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0}$$ ← continuity of energy density

Remark: Carroll then looks at $\perp$ to U^μ-direction and derives Euler's Equation from fluid mechanics.