LECTURE 8: APPLICATIONS OF COMPLETENESS AXIOM (§1.6 in Lafferriere and Man Nam.)

Th^m (1.6.1) THE ARCHIMEDEAN PROPERTY
The set $\mathbb{N}$ is unbounded above

Proof: Suppose towards $\rightarrow\leftarrow$ that $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ is nonempty, $\alpha = \sup(\mathbb{N}) \in \mathbb{R}$. Apply Prop. 1.5.1. with $\varepsilon = 1$ to see $\exists n \in \mathbb{N}$ for which $\alpha - 1 < n \le \alpha \Rightarrow n+1 > \alpha$ but $n+1 \in \mathbb{N}$ and $n+1$ is larger than $\sup(\mathbb{N})$ $\rightarrow\leftarrow$. Hence $\mathbb{N}$ not bounded above. //

Th^m (1.6.2) PROPERTIES OF $\mathbb{N} \subseteq \mathbb{R}$

(a.) for any $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $x < n$,
(b.) for any $\varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $1/n < \varepsilon$,
(c.) for any $x > 0$ and any $y \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $y < nx$
(d.) for any $x \in \mathbb{R}$, $\exists m \in \mathbb{Z}$ s.t. $m - 1 \le x < m$.

Proof: (a.) Let $x \in \mathbb{R}$. Then x is not upper bound of $\mathbb{N}$ $\therefore \exists n \in \mathbb{N}$ with $n > x$.

(b.) Let $\varepsilon > 0$ then $1/\varepsilon \in \mathbb{R}$ and so by (a) $\exists n \in \mathbb{N}$ s.t. $1/\varepsilon < n \therefore 1/n < \varepsilon$.

(c.) Apply (a.) to $y, x \in \mathbb{R}$ with $x \neq 0$ so $y/x \in \mathbb{R}$ and $\exists n \in \mathbb{N}$ with $\frac{y}{x} < n$ $\therefore y < nx$. //

(d.) See text pg. 27.

<u>Proof continued</u>

(d.) Consider $x > 0$. Let $A = \{ n \in \mathbb{N} \mid x < n \}$
then from (a.) $A \neq \emptyset$. Thus $A \subseteq \mathbb{N}$ and $A \neq \emptyset$ hence by WOP we find $\ell \in A$ s.t. $\ell \leq n \quad \forall n \in A$ and $\ell - 1 \notin A$.

Note then $x < \ell$. Either $\ell - 1 \in \mathbb{N}$ or $\ell - 1 = 0$

$$\left.\begin{array}{l} \text{If } \ell - 1 = 0 \text{ then } \ell - 1 < x. \\ \text{If } \ell - 1 \in \mathbb{N}, \text{ since } \ell - 1 \notin A \text{ we have } \ell - 1 \leq x \end{array}\right\} \quad \underbrace{\ell - 1}_{m-1} \leq x < \underbrace{\ell}_{m}$$

(set $m = \ell$)

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If $x \leq 0$ then by part (a.) there exists $N \in \mathbb{N}$ s.t. $|x| < N$.
then $-N < x < N \Rightarrow x + N > 0$. Thus apply argument given for $x > 0$ with $x \longmapsto x + N$ to find $\ell \in \mathbb{N}$ s.t. $\underbrace{\ell - 1 < x + N < \ell}$

$$\underbrace{\ell - N - 1}_{m-1} < x < \underbrace{\ell - N}_{m}$$

Example 1.6.1 $B = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\} = \{0, 1/2, 2/3, 3/4, 4/5, 5/6, \dots, n/(n+1), \dots\}$

Let $A = \sup\{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$. Prove $A = 1$

Notice $1 - \frac{1}{n} < 1$ for all $n \in \mathbb{N}$ thus 1 bounds A above.

Let $\varepsilon > 0$ and select $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$. Observe

$$-\varepsilon < \frac{-1}{n} \Rightarrow 1 - \varepsilon < 1 - \frac{1}{n} \in B$$

By Prop. 1.5.1 we find $\sup\{1 - \frac{1}{n} \mid n \in \mathbb{N}\} = 1$. (We've satisfied 1' and 2')

Th$^{\underline{m}}$ 1.6.3 - Density Property of $\mathbb{Q}$

If $x, y \in \mathbb{R}$ s.t. $x < y$ then $\exists r \in \mathbb{Q}$ s.t. $x < r < y$

Proof: Let $x, y \in \mathbb{R}$ and suppose $x < y$. Notice $y - x > 0$ hence $\exists n \in \mathbb{N}$ such that (applying Th$^{\underline{m}}$ 1.6.2 (c.) with $y \mapsto 1$, $x \mapsto y-x$)

$$1 < n(y-x)$$

Then,

$$ny = nx + ny - nx = nx + n(y-x) > nx + 1 \quad (*)$$

So applying Th$^{\underline{m}}$ 1.6.2. (d.) we select $m \in \mathbb{Z}$ such that

$$m - 1 \leq nx < m$$

using (*)

$$\Rightarrow nx < m \leq nx + 1 < ny$$

$$\Rightarrow x < m/n < y \quad //$$

(the text motivates the argument before giving it)

■ **Example 1.6.1** Let $A = \sup\{1 - \frac{1}{n} : n \in \mathbb{N}\}$. We claim that $\sup A = 1$.

We use Proposition 1.5.1. Since $1 - 1/n < 1$ for all $n \in \mathbb{N}$, we obtain condition (1'). Next, let $\varepsilon > 0$. From Theorem 1.6.2 (b) we can find $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Then

$$1 - \varepsilon < 1 - \frac{1}{n}.$$

This proves condition (2') with $a = 1 - \frac{1}{n}$ and completes the proof.

Theorem 1.6.3 — The Density Property of $\mathbb{Q}$. If x and y are two real numbers such that $x < y$, then there exists a rational number r such that

$$x < r < y.$$

Proof: We are going to prove that there exist an integer m and a positive integer n such that

$$x < m/n < y,$$

or, equivalently,

$$nx < m < ny = nx + n(y - x).$$

Since $y - x > 0$, by Theorem 1.6.2 (3), there exists $n \in \mathbb{N}$ such that $1 < n(y - x)$. Then

$$ny = nx + n(y - x) > nx + 1.$$

By Theorem 1.6.2 (4), one can choose $m \in \mathbb{Z}$ such that

$$m - 1 \leq nx < m.$$

Then $nx < m \leq nx + 1 < ny$. Therefore,

$$x < m/n < y.$$

The proof is now complete. □

We will prove in a later section (see Examples 3.4.2 and 4.3.1) that there exists a (unique) positive real number x such that $x^2 = 2$. We denote that number by $\sqrt{2}$. The following result shows, in particular, that $\mathbb{R} \neq \mathbb{Q}$.

Proposition 1.6.4 The number $\sqrt{2}$ is irrational.

Proof: Suppose, by way of contradiction, that $\sqrt{2} \in \mathbb{Q}$. Then there are integers r and s with $s \neq 0$, such that

$$\sqrt{2} = \frac{r}{s}.$$ ← all factors of r, s are reduced.

By canceling out the common factors of r and s, we may assume that r and s have no common factors.

Now, by squaring both sides of the equation above, we get

$$2 = \frac{r^2}{s^2}, \quad \Rightarrow r^2 = 2s^2 \Rightarrow r^2 \text{ is even}$$

$\Rightarrow r$ is even aka $r = 2j$ for $j \in \mathbb{Z}$

$\Rightarrow (2j)^2 = 2s^2$

$\Rightarrow 4j^2 = 2s^2 \Rightarrow s^2 = 2j^2$ ∴ s^2 even.

∴ s even.

and, hence,

$$2s^2 = r^2. \tag{1.3}$$

It follows that r^2 is an even integer. Therefore, r is an even integer (see Exercise 1.4.1). We can then write $r = 2j$ for some integer j. Hence $r^2 = 4j^2$. Substituting in (1.3), we get $s^2 = 2j^2$. Therefore, s^2 is even. We conclude as before that s is even. Thus, both r and s have a common factor, which is a contradiction. □

The next theorem shows that irrational numbers are as ubiquitous as rational numbers.

Theorem 1.6.5 Let x and y be two real numbers such that $x < y$. Then there exists an irrational number t such that

$$x < t < y.$$

Proof: Since $x < y$, one has

$$x - \sqrt{2} < y - \sqrt{2}$$

By Theorem 1.6.3, there exists a rational number r such that

$$x - \sqrt{2} < r < y - \sqrt{2}$$

This implies

$$x < r + \sqrt{2} < y.$$

Since r is rational, the number $t = r + \sqrt{2}$ is irrational (see Exercise 1.6.4) and $x < t < y$. □

Exercises

1.6.1 For each sets below determine if it is bounded above, bounded below, or both. If it is bounded above (below) find the supremum (infimum). Justify all your conclusions.

(a) $\left\{\dfrac{3n}{n+4} : n \in \mathbb{N}\right\}$

(b) $\left\{(-1)^n + \dfrac{1}{n} : n \in \mathbb{N}\right\}$

(c) $\left\{(-1)^n - \dfrac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

1.6.2 ▶ Let r be a rational number such that $0 < r < 1$. Prove that there is $n \in \mathbb{N}$ such that

$$\frac{1}{n+1} < r \leq \frac{1}{n}.$$

1.6.3 Let $x \in \mathbb{R}$. Prove that for every $n \in \mathbb{N}$, there is $r \in \mathbb{Q}$ such that $|x - r| < \dfrac{1}{n}$.