# LECTURE 9: CONVERGENCE

**Def$^{n}$/** Let $\{a_n\}$ be sequence in $\mathbb{R}$. We say $\{a_n\}$ converges to $a \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - a| < \varepsilon$. We write $a_n \longrightarrow a$ or $\lim\limits_{n\to\infty}(a_n) = a$ when the above holds.

Example 2.1.1

Let $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Let $\varepsilon > 0$ then by $\overbrace{\text{Archimedean Property}}^{A.P.}$ $\exists N \in \mathbb{N}$ s.t. $N > 1/\varepsilon$ which means $\varepsilon > \frac{1}{N}$. If $m \geq N$ then $\frac{1}{N} \geq \frac{1}{m}$

$$|a_m - 0| = \left|\frac{1}{m}\right| = \frac{1}{m} \leq \frac{1}{N} < \varepsilon \quad \therefore \lim_{m\to\infty}\left(\frac{1}{m}\right) = 0. /\!/$$

Lemma $0<a<b \Rightarrow a^\alpha < b^\alpha$ for $\alpha > 0$

Example 2.1.2

Let $\alpha > 0$ and consider $a_n = \frac{1}{n^\alpha}$.
Let $\varepsilon > 0$ then $\left(\frac{1}{\varepsilon}\right)^{1/\alpha} > 0$ thus $\exists N \in \mathbb{N}$ for which $N > \left(\frac{1}{\varepsilon}\right)^{1/\alpha}$ by A.P.
If $n \geq N$ then $n \geq N > \left(\frac{1}{\varepsilon}\right)^{1/\alpha} \Rightarrow n^\alpha > \frac{1}{\varepsilon}$. Thus,

$$\left|\frac{1}{n^\alpha} - 0\right| = \frac{1}{n^\alpha} < \frac{1}{1/\varepsilon} = \varepsilon \quad \therefore \lim_{n\to\infty}\left(\frac{1}{n^\alpha}\right) = 0.$$

**Remark:** scratch work not shown above! The backstory is to start with $|a_n - L| < \varepsilon$ and decide on $N$ (usually as fnct. of $\varepsilon$) for which $n \geq N \Rightarrow |a_n - L| < \varepsilon$.

<u>Example 2.1.3</u>: $a_n = \dfrac{3n^2+4}{2n^2+n+5} \longrightarrow \dfrac{3}{2}$ (let's prove from Def$^{\underline{n}}$ )

$$\left| \frac{3n^2+4}{2n^2+n+5} - \frac{3}{2} \right| = \left| \frac{2(3n^2+4) - 3(2n^2+n+5)}{2(2n^2+n+5)} \right| = \left| \frac{6n^2+8-6n^2-3n-15}{2(2n^2+n+5)} \right| = \left| \frac{-3n-7}{2(2n^2+n+5)} \right|$$

$$|a_n - L| = \underbrace{\frac{3n+7}{2(2n^2+n+5)} < \frac{3n+7n}{4n^2}}_{\substack{\cdot\ \text{made denominator smaller} \\ \cdot\ \text{made numerator larger}}} = \frac{10n}{4n^2} = \underbrace{\frac{5}{2n} \leq \frac{5}{2N}}_{\cdot\ n \geq N} \underbrace{< \varepsilon}_{\substack{\text{choose } N \\ \text{to make this} \\ \text{happen.}}}$$

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Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{5}{2\varepsilon} < N$ by Archimedean Property.
Suppose $n \geq N$ then observe $\frac{1}{n} \leq \frac{1}{N}$ hence $\frac{5}{2n} \leq \frac{5}{2N} < \frac{5}{2\left(\frac{5}{2\varepsilon}\right)} = \varepsilon$.

Consider,

$$\left| \frac{3n^2+4}{2n^2+n+5} - \frac{3}{2} \right| = \left| \frac{2(3n^2+4) - 3(2n^2+n+5)}{2(2n^2+n+5)} \right|$$

$$= \left| \frac{6n^2+8-6n^2-3n-15}{2(2n^2+n+5)} \right|$$

$$= \left| \frac{-3n-7}{2(2n^2+n+5)} \right|$$

$$= \frac{3n+7}{2(2n^2+n+5)} < \frac{10n}{4n^2} = \frac{5}{2n} < \varepsilon.$$

$\therefore \lim_{n\to\infty} \left( \frac{3n^2+4}{2n^2+n+5} \right) = \frac{3}{2}$ by direct argument from the Def$^{\underline{n}}$ of limit. //

Ex. 2.1.4.

$$\left|a_n - \frac{4}{3}\right| = \frac{4n-3}{3n(3n-1)} = \frac{\frac{4}{3}(3n-1) - \frac{5}{3}}{3n(3n-1)} < \frac{\frac{4}{3}\cancel{(3n-1)}}{3n\cancel{(3n-1)}} = \frac{4}{9n} \leq \frac{4}{9N} < \varepsilon$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $\frac{4}{9\varepsilon} < N \Rightarrow \frac{4}{9N} < \varepsilon$. Suppose $n \geq N$ and calculate,

$$\left|\frac{4n^2-1}{3n^2-n} - \frac{4}{3}\right| = \left|\frac{3(4n^2-1) - 4(3n^2-n)}{3(3n^2-n)}\right|$$

$$= \left|\frac{-3+4n}{3n(3n-1)}\right|$$

$$= \frac{4n-3}{3n(3n-1)} = \frac{\frac{4}{3}(3n-1) - \frac{5}{3}}{3n(3n-1)} < \frac{\frac{4}{3}\cancel{(3n-1)}}{3n\cancel{(3n-1)}} = \frac{4}{9n} < \frac{4}{9N} < \varepsilon$$

$$\therefore \lim_{n\to\infty}\left(\frac{4n^2-1}{3n^2-n}\right) = \frac{4}{3}. //$$

**Example 2.1.5.** $a_n = \dfrac{n^2+5}{4n^2+n} = \dfrac{1+5/n^2}{4+1/n} \longrightarrow \dfrac{1}{4}$ (intuition)

$$\left|a_n - \frac{1}{4}\right| = \left|\frac{n^2+5}{4n^2+n} - \frac{1}{4}\right| = \left|\frac{4(n^2+5)-(4n^2+n)}{4(4n^2+n)}\right| = \left|\frac{20-n}{4(4n^2+n)}\right|$$

We can't drop $|\cdot|$ so easily here. Notice,

$$\left|\frac{20-n}{16n^2+4n}\right| = \begin{cases} \dfrac{20-n}{16n^2+4n} & : n \leq 20 \\ \dfrac{n-20}{16n^2+4n} & : n \geq 20 \end{cases}$$

When $n \geq 20$ note $\dfrac{n-20}{16n^2+4n} < \dfrac{n}{(16n+4)n} = \dfrac{1}{16n+4} < \dfrac{1}{16n} \leq \dfrac{1}{16N} < \varepsilon$

We should build $N \geq 20$ into our choice of $N$, also $\frac{1}{16\varepsilon} < N$ hence, choose $N = \max\{20, 1/16\varepsilon\}$. Well, ... ?

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Let $\varepsilon > 0$. Notice $\frac{1}{16\varepsilon} < N'$ for some $N' \in \mathbb{N}$ by A.P. Let $N = \max\{20, N'\}$ then $N \geq 20$ and $N \geq N' > \frac{1}{16\varepsilon}$ hence $\frac{1}{16N} < \varepsilon$. Thus, if $n \geq N$ and $n \in \mathbb{N}$ we find, $n \geq N \geq 20$

$$\left|a_n - \frac{1}{4}\right| = \left|\frac{n^2+5}{4n^2+n} - \frac{1}{4}\right| = \left|\frac{4(n^2+5)-(4n^2+n)}{4(4n^2+n)}\right| = \left|\frac{20-n}{4(4n^2+n)}\right| = \frac{n-20}{(16n+4)n}$$

Thus, $\left|a_n - \frac{1}{4}\right| = \dfrac{n-20}{n(16n+4)} < \dfrac{n}{n(16n+4)} = \dfrac{1}{16n+4} < \dfrac{1}{16n} \leq \dfrac{1}{16N} < \varepsilon.$

Thus $a_n \longrightarrow \frac{1}{4}$ as $n \longrightarrow \infty$. //

**Lemma 2.1.2:** Let $l \geq 0$. If $l < \varepsilon$ for all $\varepsilon > 0$, then $l = 0$.

Proof: Suppose $l \geq 0$ and $l < \varepsilon \ \forall \varepsilon > 0$. If $l > 0$ then $\varepsilon = l/2 > 0$ hence $l < l/2$ which is absurd. Therefore, $l = 0$.//

**Th$^{m}$ 2.1.3:** A convergent sequence $\{a_n\}$ has at most one limit.

Proof: Suppose $a_n \longrightarrow a$ and $a_n \longrightarrow b$ as $n \longrightarrow \infty$. Let $\varepsilon > 0$ then $\varepsilon/2 > 0$ hence $\exists N_a, N_b \in \mathbb{N}$ for which $n \geq N_a \Rightarrow |a_n - a| < \varepsilon/2$ and $n \geq N_b \Rightarrow |a_n - b| < \varepsilon/2$

Let $n \geq \max\{N_a, N_b\}$ then,

$$0 \leq |a-b| = |a_n - b - (a_n - a)| \leq |a_n - b| + |-(a_n - a)| = |a_n - b| + |a_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

thus $l = |a-b| < \varepsilon \ \forall \varepsilon > 0$ and we find $|a-b| = 0$ by Lemma. Thus $a = b$.//

**Lemma 2.1.4:** Given $a, b \in \mathbb{R}$, $a \leq b \iff a < b + \varepsilon \ \forall \varepsilon > 0$

Proof: $\Rightarrow$) Suppose $a \leq b$ then $a - b \leq 0 < \varepsilon \ \forall \varepsilon > 0$.

$\Leftarrow$) Suppose $a < b + \varepsilon \ \forall \varepsilon > 0$. Then $a - b < \varepsilon \ \forall \varepsilon > 0$. If $a - b < 0$ then $a < b$ thus $a \leq b$ is true. If $a - b \geq 0$ then by Lemma 2.1.2 identify $l = a - b \geq 0$ with $l = a - b < \varepsilon \ \forall \varepsilon > 0$ hence $l = a - b = 0$ and we find $a = b$ thus $a \leq b$ is true.//

**Th$^{\underline{m}}$ 2.1.5:** Suppose $\{a_n\}$ and $\{b_n\}$ converge to $a$ & $b$ respectively and $a_n \le b_n$ for all $n \in \mathbb{N}$ then $a \le b$. (COMPARISON THEOREM)

**Proof:** Suppose $a_n \to a$ and $b_n \to b$ where $a_n \le b_n$ $\forall n \in \mathbb{N}$.
Let $\varepsilon > 0$ then $\varepsilon/2 > 0$ hence $\exists N_a, N_b \in \mathbb{N}$ for which

① $|a_n - a| < \varepsilon/2$ whenever $n \ge N_a$

② $n \ge N_b \Rightarrow |b_n - b| < \varepsilon/2$

Thus $n \ge \max\{N_a, N_b\}$ we have

$$-\varepsilon/2 < a_n - a < \varepsilon/2 \quad \text{and} \quad -\varepsilon/2 < b_n - b < \varepsilon/2$$

$$\therefore \quad a - \varepsilon/2 < a_n < a + \varepsilon/2 \quad \text{and} \quad b - \varepsilon/2 < b_n < b + \varepsilon/2$$

Therefore, $a - \varepsilon/2 < a_n \le b_n < b + \varepsilon/2 \Rightarrow \underbrace{a < b + \varepsilon \quad \therefore \quad a \le b}_{\text{By Lemma 2.1.4.}}$ //

**Th$^{\underline{m}}$ 2.1.6: (SQUEEZE THEOREM)**
Suppose $a_n \le b_n \le c_n$ and $a_n \to a$ and $c_n \to c$ with $a = c$ then $b_n \to a$.

**Proof:** Let $\varepsilon > 0$. Since $a_n \to a$, $\exists N_a \in \mathbb{N}$ s.t. $a - \varepsilon < a_n < a + \varepsilon$ when $n \ge N_a$
Likewise since $c_n \to a$, $\exists N_c \in \mathbb{N}$ s.t. $n \ge N_c$ implies $a - \varepsilon < c_n < a + \varepsilon$. Let $N = \max\{N_a, N_c\}$
if $n \ge N$ then observe

$$a - \varepsilon < a_n \le b_n \le c_n < a + \varepsilon \Rightarrow -\varepsilon < b_n - a < \varepsilon$$
$$\Rightarrow |b_n - a| < \varepsilon$$
$$\therefore \quad b_n \to a. //$$

**Def$^n$/** A sequence $\{a_n\}$ is bounded above if $\{a_n \mid n \in \mathbb{N}\}$ is bounded above. Likewise a sequence is bounded below if $\underbrace{\{a_n \mid n \in \mathbb{N}\}}_{\text{range of the sequence.}}$ is bounded below.
$\{a_n\}$ is bounded if it is bounded above & below.

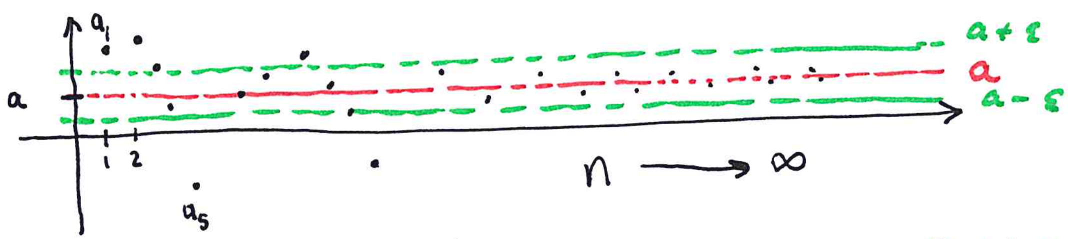
$a : \mathbb{N} \longrightarrow \mathbb{R}$

$a(n) = n$ or $n \longmapsto a_n$

range $(a) = \{a_n \mid n \in \mathbb{N}\} = \{a_1, a_2, \dots\}$

Remark: A sequence $\{a_n\}$ is bounded iff $\exists M > 0$ such that $|a_n| < M$ $\forall n \in \mathbb{N}$.

## Th$^m$ 2.1.7: A CONVERGENT SEQUENCE IS BOUNDED




**Proof**: Suppose $a_n \longrightarrow a$. Let $\varepsilon = 1 > 0$ then $\exists N \in \mathbb{N}$ s.t. $n \geq N$ implies $|a_n - a| < 1$,

$$-1 < a_n - a < 1 \quad \Rightarrow \quad a - 1 < a_n < a + 1$$

thus $|a_n| < \max\{|a-1|, |a+1|\}$ for $n \geq N$. Let

$$M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |a-1|, |a+1|\}$$

then clearly $|a_n| \leq M$ $\forall n \in \mathbb{N}$. Thus $\{a_n\}$ is bounded. //

Remark: $|a_n - a| < 1$ and $|a_n| - |a| \leq \big||a_n| - |a|\big| \leq |a_n - a| < 1 \Rightarrow |a_n| < 1 + |a|$

can use instead of $|a-1|, |a+1|$ if you prefer.

**Def$^n$/** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real #'s. The sequence $\{b_n\}_{n=1}^{\infty}$ is called a <u>subsequence</u> of $\{a_n\}_{n=1}^{\infty}$ if $\exists$ sequence of increasing positive integers $n_1 < n_2 < n_3 < \dots$ such that $b_k = a_{n_k}$ for each $k \in \mathbb{N}$.

<u>Example 2.1.6</u>: $a_n = (-1)^n$ has subsequences $\{a_{2k}\} = \{(-1)^{2k}\} = \{1, 1, \dots\}$ & $\{a_{2k-1}\}_{k=1}^{\infty} = \{-1, -1, -1, \dots\}$ that is $\underbrace{a_{2k} = (-1)^{2k} = 1}$ & $a_{2k+1} = (-1)^{2k+1} = -1$

$n_k = 2k$ $\quad$ $n_k = 2k+1$

$b_k = 1$ $\quad$ $b_k = -1$

<u>Lemma 2.1.8</u>: Let $\{n_k\}$ be sequence of positive integers with $n_1 < n_2 < n_3 < \dots$ then $n_k \geq k$ $\forall k \in \mathbb{N}$.

<u>Proof</u>: by induction on $k$. Observe $n_1 \geq 1$ since $\{n_k\}$ is sequence in $\mathbb{N}$. Suppose inductively $n_k \geq k$ for some $k \in \mathbb{N}$. Consider $n_{k+1} > n_k$ and thus $n_{k+1} > n_k \geq k$ but $n_{k+1}, n_k \in \mathbb{N}$ thus

$n_{k+1} - 1 \geq n_k \geq k \Rightarrow n_{k+1} \geq k+1$ hence Lemma true by PMI on $k$. //

**Th$^{\underline{m}}$ (2.1.9):** If $\{a_n\}$ converges to $a$ then any subsequence $\{a_{n_k}\}$ of $\{a_n\}$ likewise converges to $a$.

Proof: Suppose $\{a_n\}$ converges to $a$, and has subsequence $\{a_{n_k}\}$. Let $\varepsilon > 0$ then $\exists N \in \mathbb{N}$ for which $n \geq N$ implies $|a_n - a| < \varepsilon$. But, for any $k \geq N$ we have $\underbrace{n_k \geq k}_{\text{using Lemma 2.1.8.}}$ we likewise obtain $|a_{n_k} - a| < \varepsilon \therefore \lim_{k \to \infty}(a_{n_k}) = a$. //

Remark: So... if $\exists$ a subsequence of $\{a_n\}$ which does not converge, OR if $\exists$ two subsequences which converge distinctly $a_{n_k} \to a \neq a'$, $a_{m_j} \to a' \neq a$ then $\{a_n\}$ diverges. (is divergent)

Example 2.1.7

$$a_n = (-1)^n \quad \begin{matrix} \nearrow & a_{2k} = 1 \longrightarrow 1 \\ \searrow & a_{2k-1} = -1 \longrightarrow -1 \end{matrix} \quad \therefore \{a_n\} = \{(-1)^n\} \text{ divergent.}$$

CAUTION: $\{a_n\}_{n=k_o}^{\infty} = \{a_{k_o}, a_{k_o+1}, \dots\}$ also considered a sequence beyond this point. Indeed $a_n = \frac{1}{(n-1)(n-2)}$ will have domain $\{3, 4, 5, \dots\}$ by convention of most authors. better to write $\left\{\frac{1}{(n-1)(n-2)}\right\}_{n=3}^{\infty}$ for specificity...