Lecture 9: Lagrangian Mechanics

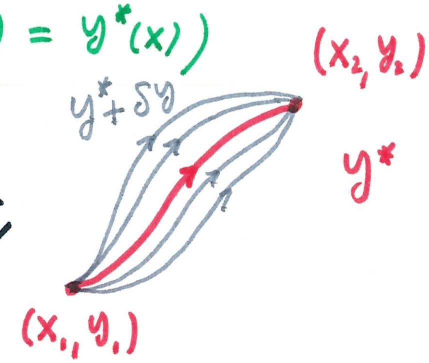
We should take a few minutes to discuss <u>variational</u> <u>calculus</u>. The problem of variational calculus is to find a <u>function</u> which optimizes a given <u>functional</u> J with respect to some family of appropriate functions. Typically involves integrating a <u>Lagrangian</u>

$$J[y] = \int_{x_1}^{x_2} f(y, y', x)\,dx$$

Here $y \in \mathcal{F}_0$ which is the set of differentiable functions whose graphs contain (x_1, y_1) and (x_2, y_2). If $y^* \in \mathcal{F}_0$ and we introduce another function η for which $\eta(x_1) = \eta(x_2) = 0$ then we may study a whole family of functions in $\mathcal{F}_0$ parametrized by α

$$y(x, \alpha) = y(x, 0) + \alpha\,\eta(x) \quad (\text{where } y(x,0) = y^*(x))$$

$$\boxed{\delta y = \alpha\,\eta(x)} \leftarrow \text{variation of } y$$



Then we may restrict the functional to this class of paths,

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), y(x,\alpha)', x)\,dx$$

<u>Claim</u>: if y^* optimizes J then we should expect $\left[\frac{\partial J}{\partial \alpha}\right]\Big|_{\alpha=0} = 0$

Keep in mind $y(x,\alpha) = y(x,0) + \alpha\eta(x)$ hence $y(x,\alpha)' = y(x,0)' + \alpha\eta'(x)$

therefore $\frac{\partial y}{\partial \alpha} = \eta$ whereas $\frac{\partial y'}{\partial \alpha} = \eta' = \frac{d\eta}{dx}$. Apply multivariate chain rule and assume $\frac{\partial}{\partial\alpha}\int = \int\frac{\partial}{\partial\alpha}$ for what follows:

$$\frac{\partial J(\alpha)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left[\int_{x_1}^{x_2} f(y(x,\alpha), y(x,\alpha)', x)\,dx\right]$$

$$= \int_{x_1}^{x_2}\frac{\partial}{\partial\alpha}\left[f(y(x,\alpha), y(x,\alpha)', x)\right]dx$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial\alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial\alpha} + \frac{\partial f}{\partial x}\cancelto{0}{\frac{\partial x}{\partial\alpha}}\right)dx$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right)dx$$

$$\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\eta\right] = \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta + \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\eta\right] - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta\right)dx$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\right)\eta(x)\,dx + \underbrace{\int_{x_1}^{x_2}\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\eta\right]dx}_{\frac{\partial f}{\partial y'}\eta(x_2) - \frac{\partial f}{\partial y'}\eta(x_1) = 0}$$

But, η was arbitrary. It follows we need the Euler-Lagrange Equation to hold for the path which optimizes the functional $J[y] = \int_{x_1}^{x_2} f(y, y', x)\,dx$

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0}$$

Euler-Lagrange Eq $\underline{n}$.

The calculus we saw on 2 can be written more formally in terms of the <u>variation</u>. Physicists speak of "taking the variation"

$$\delta J = \delta\left[\int_{x_1}^{x_2} f(y, y', x)\, dx\right]$$

$$= \int_{x_1}^{x_2} \delta f(y, y', x)\, dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + \frac{\partial f}{\partial x}\delta x\right) dx$$

$$\delta y' = \left(\delta \frac{d}{dx}\right) y = \frac{d}{dx}(\delta y)$$

$$\delta x = 0$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)\right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\delta y + \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\delta y\right] - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\delta y\right) dx$$

$$= \underbrace{\cancel{\frac{\partial f}{\partial y'}\delta y\Big|_{x_1}^{x_2}}}_{0} + \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\right)\delta y\, dx = 0$$

Since $\delta J = 0$ for arbitrary variations of δy we find once again

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0}$$

LAGRANGIAN MECHANICS:

Let $y_1, y_2, \dots, y_n$ be coordinates describing a physical system and $\dot{y}_1, \dot{y}_2, \dots, \dot{y}_n$ their corresponding velocities. Let T be the kinetic energy of the system and U the potential energy then the Lagrangian is given by $L = T - U$ and the action

$$S = \int L \, dt$$

Then taking the variation of S and supposing $\delta S = 0$ is Hamilton's principle of least action. The variations of $y_1, y_2, \dots, y_n$ are independent in the sense each gives a separate Euler-Lagrange Eq:

$$\delta S = \int \left(\sum_{i=1}^{n} \left(\frac{\partial L}{\partial y_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right)\right) \delta y_i \right) dt = 0$$

Euler Lagrange Eq$^{\underline{s}}$:

$$\boxed{\begin{aligned} \frac{\partial L}{\partial y_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) &= 0 \\ \frac{\partial L}{\partial y_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_2}\right) &= 0 \\ &\vdots \\ \frac{\partial L}{\partial y_n} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_n}\right) &= 0 \end{aligned}}$$

Remark: apparently we're treating y_j and $\dot{y}_j$ as independent variables. This can be formalized with jet-space...

E1 free particle with coordinates (x, y, z) and mass m

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial \dot{y}} = m\dot{y} \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \qquad \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \qquad \frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0$$

$$\boxed{m\ddot{x} = 0} \qquad \boxed{m\ddot{y} = 0} \qquad \boxed{m\ddot{z} = 0}$$

Remark: it's nice to use EL-Eq^ns as:

$$\frac{\partial L}{\partial y_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_j}\right)$$

These we can solve by twice integrating,

$$\boxed{x = x_o + tV_x} \qquad \boxed{y = y_o + tV_y} \qquad \boxed{z = z_o + tV_z}$$

E2 three dimensional spring anchored at origin with spring constant k pulls/pushes on mass m.

$$\vec{F}_{spring} = -k\vec{r} = \langle -kx, -ky, -kz \rangle = -\nabla\Big(\underbrace{\tfrac{1}{2}k(x^2+y^2+z^2)}_{PE_{spring} = U}\Big)$$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{1}{2}k(x^2+y^2+z^2)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial \dot{y}} = m\dot{y} \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial L}{\partial x} = -kx \qquad \frac{\partial L}{\partial y} = -ky \qquad \frac{\partial L}{\partial z} = -kz$$

$$\frac{d}{dt}(m\dot{x}) = -kx$$

$$\frac{d}{dt}(m\dot{y}) = -ky$$

$$\frac{d}{dt}(m\dot{z}) = -kz$$

E2 continued

$$\left.\begin{aligned} m\ddot{x} &= -kx \\ m\ddot{y} &= -ky \\ m\ddot{z} &= -kz \end{aligned}\right\}$$ Euler-Lagrange Eq^{ns} are precisely Newton's 2^{nd} Law in this case and $\boxed{E1}$.

$\boxed{E3}$ Suppose a mass m has PE given a function of $r = \sqrt{x^2+y^2}$ then polar coordinates r, θ are a nice choice.

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

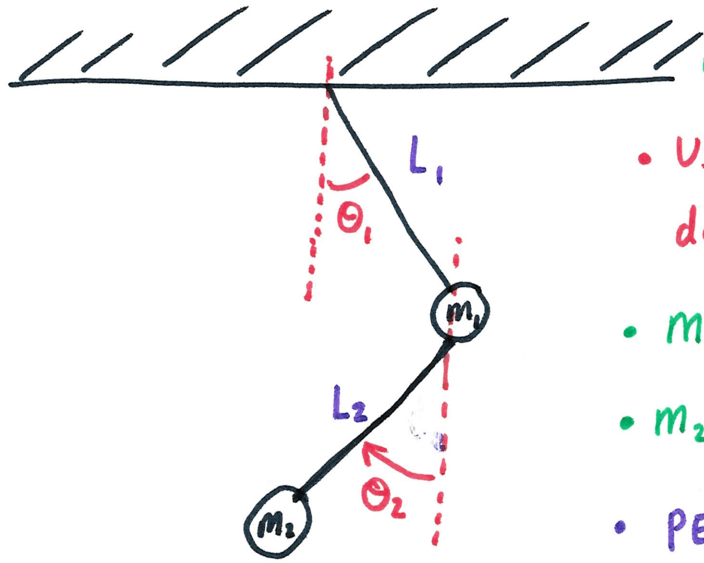
$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \Rightarrow 0 = \frac{d}{dt}\left[mr^2\dot{\theta}\right] \Rightarrow \underbrace{\boxed{\ell_0 = mr^2\dot{\theta}}}_{\text{conservation of angular momentum}}.$$

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)\right) = m\ddot{r}$$

$$\underbrace{\boxed{mr\dot{\theta}^2 - \frac{dU}{dr} = m\ddot{r}}}_{\text{Newton's } 2^{nd} \text{ Law for radial coordinate}}$$

E4 Double Pendulum



$y = 0$

- Use θ_1 & θ_2 as coordinates to describe this double pendulum
- m_1 is at $y_1 = -L_1 \cos\theta_1$
- m_2 is at $y_2 = -L_1\cos\theta_1 - L_2 \cos\theta_2$
- PE = mgy for m at y close to Earth surface

$$L = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \underbrace{\frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2}_{} - m_1 g y_1 - m_2 g y_2$$

might be wrong...
need to check, YEP, it's missing the $L_1^2 \dot{\theta}_1^2$

$$x_2 = L_1 \sin\theta_1 - L_2 \sin\theta_2 \qquad \dot{x}_2 = L_1 \cos\theta_1 \dot{\theta}_1 - L_2 \sin\theta_2 \dot{\theta}_2$$

$$y_2 = -L_1 \cos\theta_1 - L_2 \cos\theta_2 \qquad \dot{y}_2 = +L_1 \sin\theta_1 \dot{\theta}_1 + L_2 \cos\theta_2 \dot{\theta}_2$$

$$\dot{x}_2^2 + \dot{y}_2^2 = (L_1\cos\theta_1\dot{\theta}_1 - L_2 \sin\theta_2 \dot{\theta}_2)^2 + (L_1 \sin\theta_1 \dot{\theta}_1 + L_2 \cos\theta_2 \dot{\theta}_2)^2$$

$$= L_1^2(\cos^2\theta_1 + \sin^2\theta_1)\dot{\theta}_1^2 + L_2^2(\sin^2\theta_2 + \cos^2\theta_2)\dot{\theta}_2^2 + \ldots$$

$$= L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 - 2L_1L_2\cos\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2 + 2L_1L_2\sin\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2$$

$$\boxed{L = \frac{m_1}{2} L_1^2 \dot{\theta}_1^2 + \frac{m_2}{2}(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2) + m_1 g L_1 \cos\theta_1 + m_2 g (L_1\cos\theta_1 + L_2\cos\theta_2) + - m_2 L_1 L_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2}$$

E4 continued

$+\left(-m_2 L_1 L_2 \cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right.$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = -m_1 g L_1 \sin\theta_1 - m_2 g L_1 \sin\theta_1 - \frac{d}{dt}\left[m_1 L_1^2 \dot{\theta}_1 + m_2 L_1^2 \dot{\theta}_1\right] = 0$$

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \underbrace{-m_2 g L \sin\theta_2 - \frac{d}{dt}\left[m_2 L_2^2 \dot{\theta}_2\right]}_{\text{missing couple terms from the term just discovered mid LECTURE.}} = 0$$

$+\frac{d}{dt}\left(-m_2 L_1 L_2 \sin(\theta_1-\theta_2)\dot{\theta}_1\right)$

$\boxed{\text{E5}}$ Free particle described with spherical coordinates

$$x = r\sin\theta\cos\varphi \qquad \dot{x} = \dot{r}\sin\theta\cos\varphi + r\cos\theta\cos\varphi\,\dot{\theta} - r\sin\theta\sin\varphi\,\dot{\varphi}$$

$$y = r\sin\theta\sin\varphi \qquad \dot{y} = \dot{r}\sin\theta\sin\varphi + r\cos\theta\sin\varphi\,\dot{\theta} + r\sin\theta\cos\varphi\,\dot{\varphi}$$

$$z = r\cos\theta \qquad \dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2 \quad \leftarrow \text{speed squared in spherical coordinates.}$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)$$

$$\frac{\partial L}{\partial r} = \frac{d}{dt}(m\dot{r}) \longrightarrow mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\varphi}^2 = m\ddot{r}$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \longrightarrow mr^2\sin\theta\cos\theta\,\dot{\varphi}^2 = mr^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) \longrightarrow 0 = \frac{d}{dt}\left(mr^2\sin^2\theta\,\dot{\varphi}\right)$$

$\boxed{E6}$ particle constrained to cylinder $r = R$ where $x = r\cos\theta$ and $y = r\sin\theta$. Describe motion

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \lambda(r-R) + \frac{1}{2}m\dot{z}^2$$

$$\frac{\partial L}{\partial z} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) \rightarrow \underline{0 = m\ddot{z}}\,*$$

$$\frac{\partial L}{\partial \lambda} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) \rightarrow -(r-R) = 0 \quad \therefore \quad \underline{r = R}.$$

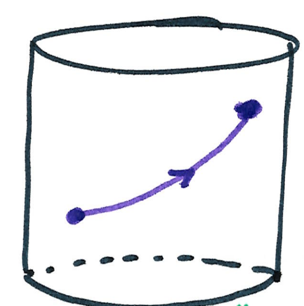
$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) \rightarrow mr\dot{\theta}^2 + \lambda = \frac{d}{dt}(m\dot{r}) \rightarrow m\ddot{r} = mr\dot{\theta}^2 + \lambda$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \rightarrow 0 = \frac{d}{dt}(mr^2\dot{\theta}) \Rightarrow L_o = mr^2\dot{\theta} \text{ is } \underline{\text{conserved}}$$

Since $r = R$ we have $\dot{r} = 0$ and $\ddot{r} = 0$ thus $\lambda = -mr\dot{\theta}^2 = \frac{-mv^2}{R}$ setting $r = R$ and $v = R\dot{\theta}$. Identify λ as the force required to keep m from flying off the cylinder.

$$\dot{\theta} = \frac{L_o}{mR^2} \equiv \omega_o \quad \text{yields} \quad \theta = \theta_o + \omega_o t$$

$$m\ddot{z} = 0 \quad \text{yields} \quad z = z_o + V_z t$$



$\vec{r}(t) = (R\cos(\theta_o + \omega_o t), R\sin(\theta_o + \omega_o t), z_o + V_z t)$ ← "geodesic" on cylinder

$\omega_o = 0$, $\vec{r}(t) = (R\cos\theta_o, R\sin\theta_o, z_o + V_z t)$ ← vertical line

$V_z = 0$, $\vec{r}(t) = (R\cos(\theta_o + \omega_o t), R\sin(\theta_o + \omega_o t), z_o)$ ← CIRCLE