Homework 10: Eigenvalues & Eigenvectors, some complex

§5.1 # 66, 69, 73 // §5.2 # 48

§5.1#66 Prove that $A \in \mathbb{R}^{n\times n}$ is invertible iff $\lambda = 0$ is not an eigenvalue.

Proof: Assume A^{-1} exists then $Ax = 0 \Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$. ~~Therefore~~
Therefore, $\nexists\, v \neq 0$ such that $Av = 0$ hence 0 is not an e-value.
Conversely, assume zero is <u>not</u> an e-value for A. This means that $\text{Null}(A) = \{0\}$ hence $\nu = 0$. We know $\dim(\text{col}(A)) + \nu = n$ hence $\dim(\text{col}(A)) = n$ so L_A is onto $\mathbb{R}^{n\times 1}$ $(L_A(x) = Ax)$. Likewise, $\text{Kernel}(L_A) = \{0\}$ since $Ax = 0 \Rightarrow x = 0$. Thus L_A is a bijection and so L_A^{-1} exists & $[L_A^{-1}] = A^{-1}$.

Convoluted

Or, $Ax = 0 \Rightarrow x = 0$ thus A^{-1} exists.
(there are dozens of reasonable arguments given our dimension theorems etc...)

§5.1#69 Prove that if λ is an e-value of A then λ^2 is an e-value of A^2

Suppose A has e-vector V with e-value λ, this means $AV = \lambda V$.
Notice $A^2V = AAV = A\lambda V = \lambda AV = \lambda\lambda V = \lambda^2 V$ $\therefore$ λ^2 is e-value of A^2 with e-vector V.

§5.1#73 Let V_1 and V_2 be e-vectors of a linear operator T on $\mathbb{R}^{n\times 1}$, and let λ_1 and λ_2 be their respective e-values. Prove that if $\lambda_1 \neq \lambda_2$ then $\{V_1, V_2\}$ is linearly independent

Suppose $V_1 = kV_2$ towards a contradiction. Notice that

$$
\begin{aligned}
T(V_1) = T(kV_2) &\Rightarrow \lambda_1 V_1 = kT(V_2) = k\lambda_2 V_2 \\
&\Rightarrow k\lambda_1 V_2 = k\lambda_2 V_2 \\
&\Rightarrow k(\lambda_1 - \lambda_2)V_2 = 0 \\
&\Rightarrow \text{either } k = 0 \text{ or } \lambda_1 = \lambda_2 \text{ or } V_2 = 0
\end{aligned}
$$

But, $\lambda_1 \neq \lambda_2$ & $V_2 \neq 0$ $\Rightarrow$ $\nexists\, k \neq 0$ such that $V_1 = kV_2$
$\Rightarrow$ $\{V_1, V_2\}$ is L.I. //

§5.2 #48 Find e-vectors and e-values for $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$
(find basis for e-space, in this case find an example e-vector

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{pmatrix}$$
$$= (\lambda - 2)^2 + 1$$
$$= 0 \quad \Rightarrow \quad \underline{\lambda = 2 \pm i}.$$

Find $\vec{u} = [u, v]^T \in \mathbb{C}^{2\times 1}$ such that $\lambda = 2+i$ meaning that $\vec{u}$ solves $(A - (2+i)I)\vec{u} = 0$;

$$\begin{bmatrix} 2-(2+i) & -1 \\ 1 & 2-(2+i) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, $-iu - v = 0$ and $u - iv = 0$ which are actually the same eq^{n} since $iv = \underline{u} \Rightarrow -v = iu \Rightarrow \underline{v = -iu}$.
Choose $u = 1$ to obtain $\boxed{\vec{u} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}}$

this decomposition into
$\vec{u} = Re(\vec{u}) + i\,Im(\vec{u})$
is important for applications.

Remark: I suspect my proofs for $\Leftarrow$ of #66 and #73 are not very clear. My apologies. Please ask me if you need them explained a different way.