Homework 11: diagonalization & eigenbases of linear operators

§5.3 # 14, 22, 82 // §5.4 # 9, 22, 26

$\boxed{\text{§5.3\#14}}$ $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ try to diagonalize A, if not possible explain why.

Goal: find e-basis for A if possible. (could fail due to complex e-values or because we only have 1 e-vector in repeated real e-value case)

$$\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} = (\lambda-4)(\lambda+1) - 6 = \lambda^2 - 3\lambda - 10 = 0$$

Hence $\det(A-\lambda I) = (\lambda-5)(\lambda+2) = 0 \Rightarrow \underline{\lambda_1 = 5 \ \& \ \lambda_2 = -2}$.

Find $\vec{u}_1$ with $(A - 5I)\vec{u}_1 = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 3u - v = 0 \Rightarrow v = 3u \longrightarrow \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$.

Likewise, $(A + 2I)\vec{u}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow u + 2v = 0, \ u = -2v \longrightarrow \underline{\vec{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$.

$$P = [\vec{u}_1 | \vec{u}_2] = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \longrightarrow P^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

Calculate then,

$$\begin{aligned} P^{-1}AP &= \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 15 & -2 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 35 & 0 \\ 0 & -14 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

no surprises here. we find e-values of A on the diagonal as expected from our discussions in lecture.

§5.3 # 22 Let $A = \begin{bmatrix} 5 & 51 \\ -3 & 11 \end{bmatrix}$ find D, P such that $A = PDP^{-1}$ for some invertible matrix P and diagonal matrix D, expect complex #'s in both D and P

$$\det(A - \lambda I) = \det\begin{bmatrix} 5-\lambda & 51 \\ -3 & 11-\lambda \end{bmatrix}$$
$$= (\lambda-5)(\lambda-11) + 153$$
$$= \lambda^2 - 16\lambda + 208$$
$$= (\lambda-8)^2 + 144 = 0 \Rightarrow \underline{\lambda = 8 \pm 12i}.$$

$\lambda = 8+12i$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5-(8+12i) & 51 \\ -3 & 11-(8+12i) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -3-12i & 51 \\ -3 & 3-12i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \underbrace{(-3-12i)u + 51v = 0}_{\text{Same as } 2^{nd} \text{ row's eq}^n.}$$

Let $u = -3+12i$ and notice $(-3-12i)(-3+12i) = 9+144 = 153$ hence $153 + 51v = 0 \Rightarrow \underline{v = 3}$. Therefore, $\underline{\vec{u}_1 = \begin{bmatrix} -3+12i \\ 3 \end{bmatrix}}$.

We can divide by 3, $\vec{u}_1 = \begin{bmatrix} -1+4i \\ 1 \end{bmatrix}$ (easier to work with)

then $\lambda^* = 8-12i$ has $\vec{u}_2 = \begin{bmatrix} -1-4i \\ 1 \end{bmatrix}$ so construct $P = [\vec{u}_1 | \vec{u}_2]$

$$P = \left[\begin{array}{c|c} -1+4i & -1-4i \\ 1 & 1 \end{array}\right] \longrightarrow P^{-1} = \frac{1}{-1-4i+1-4i}\left[\begin{array}{c|c} 1 & 1+4i \\ -1 & -1+4i \end{array}\right] = \frac{i}{8}\left[\begin{array}{c|c} 1 & 1+4i \\ -1 & -1+4 \end{array}\right]$$

Calculate, for $D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix}$,

$$PDP^{-1} = \left[\begin{array}{c|c} -1+4i & -1-4i \\ 1 & 1 \end{array}\right]\left[\begin{array}{c|c} 8+12i & 0 \\ 0 & 8-12i \end{array}\right]P^{-1}$$
$$= \frac{i}{8}\left[\begin{array}{c|c} (-1+4i)(8+12i) & (1+4i)(12i-8) \\ 8+12i & 8-12i \end{array}\right]\left[\begin{array}{c|c} 1 & 1+4i \\ -1 & -1+4i \end{array}\right]$$
$$= \frac{i}{8}\left[\begin{array}{c|c} (-1+4i)(8+12i) - (1+4i)(12i-8) & (-1+4i)(8+12i)(1+4i) + (1+4i)(12i-8)(4i- \\ (8+12i) - (8-12i) & (8+12i)(1+4i) + (8-12i)(4i-1) \end{array}\right]$$

$$= \underline{\begin{bmatrix} 5 & 51 \\ -3 & 11 \end{bmatrix}}.$$

$\boxed{\S 5.3 \# 82}$ If A is diagonalizable and A^{-1} exists prove that A^{-1} is diagonalizable

Since A diagonalizable $\Rightarrow \exists P \in \mathbb{R}^{n\times n}$ such that $P^{-1}AP = D$.
Let $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ note that $D^{-1} = \begin{bmatrix} 1/\lambda_1 & & & \\ & 1/\lambda_2 & & \\ & & \ddots & \\ & & & 1/\lambda_n \end{bmatrix}$ and

$\lambda_j \neq 0$ for $j = 1, 2, 3, \dots, n$ since $\det(P^{-1}AP) = \det(D)$

and $\det(P^{-1}AP) = \det(PP^{-1}A) = \det(A) \neq 0$ $\therefore$ zero not an e-value of A. Thus, using our little Th^{m} from Problem Set,

$$(P^{-1}AP)^{-1} = D^{-1} \Rightarrow P^{-1}A^{-1}(P^{-1})^{-1} = D^{-1}$$
$$\Rightarrow P^{-1}A^{-1}P = D^{-1} = \underbrace{\begin{bmatrix} 1/\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_n \end{bmatrix}}_{\text{diagonal}}$$

$\therefore$ $\underline{A^{-1} \text{ diagonalized}}$

$\boxed{\S 5.4 \# 9}$ Let $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 7x_1 - 6x_2 \\ 9x_1 - 7x_2 \end{bmatrix}$ find e-values & e-vectors

Note $T(v) = Av$ if $A = \begin{bmatrix} 7 & -6 \\ 9 & -7 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 7-\lambda & -6 \\ 9 & -7-\lambda \end{pmatrix} = (\lambda+7)(\lambda-7) + 54 = \underline{\lambda^2 + 5 = 0}$$

Thus $\lambda = \pm i\sqrt{5}$ are the (complex) e-values.

$\boxed{\lambda = i\sqrt{5}}$ $\begin{bmatrix} 7 - i\sqrt{5} & -6 \\ 9 & -7 - i\sqrt{5} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \underbrace{(7 - i\sqrt{5})u - 6v = 0}_{}$

same as 2^{nd} eq^{n}. (How do I know this?)

Let $u = 1$ then $(7 - i\sqrt{5}) = 6v$
thus $v = \frac{1}{6}(7 - i\sqrt{5})$ $\therefore$ $\boxed{\vec{u}_1 = \begin{bmatrix} 1 \\ \frac{1}{6}(7 - i\sqrt{5}) \end{bmatrix}}$ $\leftarrow \lambda = i\sqrt{5}$

Remark: the sol^{n} to the text's problem was just "no, only imaginary e-values $\therefore$ cannot be diagonalized over $\mathbb{R}^{2\times 1}$" (complex case)

likewise $\boxed{\vec{u}_2 = \begin{bmatrix} 1 \\ \frac{1}{6}(7 + i\sqrt{5}) \end{bmatrix} \text{ for } \lambda = -i\sqrt{5}}$

(Th^{m} says if $Av = \lambda v$ then $Av^* = \lambda^* v^*$)

<u>Remark:</u> the sol^{n} just given for §5.4 # 9 was not needed according to the text's problem statement. I'm leaving it because I do want you to know how to find complex e-vectors and e-values (even though they don't give a diagonalization within the confines of $\mathbb{R}^{n\times 1}$)

§5.4 # 22 Diagonalize

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + 3x_2 \\ -4x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

by find e-basis for T if possible

$$\det([T]-\lambda I) = \det\begin{pmatrix} -1-\lambda & 3 \\ -4 & 6-\lambda \end{pmatrix} = (\lambda-6)(\lambda+1)+12$$
$$= \lambda^2 - 5\lambda + 6$$
$$= (\lambda-3)(\lambda-2) \longrightarrow \underline{\lambda_1 = 3,\ \lambda_2 = 2}$$

$\therefore$ it <u>will</u> be diagonalizable. (how do I know?) w/o further calculatio

$\underline{\lambda_1 = 3}$
$$\begin{bmatrix} -4 & 3 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow -4u + 3v = 0$$

let $\underline{v=4}$ then $4u = 12 \therefore \underline{u = 3}$

$$\underline{\vec{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}}$$

$\underline{\lambda_2 = 2}$
$$\begin{bmatrix} -3 & 3 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow u = v \therefore \underline{\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Let $\boxed{\beta = \{\vec{u}_1, \vec{u}_2\}}$ then $[\beta] = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$ & $[\beta]^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -1 \\ -4 & 3 \end{bmatrix}$ and we calculate

$$\left.\begin{aligned} [T]_\beta &= [\beta]^{-1}[T][\beta] \\ &= \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}. // \end{aligned}\right\}$$

see, it works!

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§5.4 #26 Find diagonalizing e-basis for T if possible. Let $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 4 & -5 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\det(A-\lambda I) = \det\begin{bmatrix} 4-\lambda & -5 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix} = (4-\lambda)(-1-\lambda)(-1-\lambda) = (\lambda+1)^2(4-\lambda)$$

$\Rightarrow$ $\lambda_1 = \lambda_2 = -1$. $\lambda_3 = 4$.

$\lambda_1 = \lambda_2 = -1$ Find $(A+I)\vec{u}_{1,2} = 0$

$$\begin{bmatrix} 3 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow 3u - 5v = 0$$

We find w is free and $u = \frac{5}{3}v$ so

$$\vec{u}_{1,2} = \begin{bmatrix} (5/3)v \\ v \\ w \end{bmatrix} = (v/3)\begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} + w\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Can use $\vec{u}_1 = [5,3,0]^T$ & $\vec{u}_2 = [0,0,1]^T$ for e-basis of $W_{\lambda=-1}$

$\lambda_3 = 4$ Find $(A-4I)\vec{u}_3 = 0$

$$\begin{bmatrix} 0 & -5 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \longrightarrow 5v = 0 \\ \longrightarrow 6w = 0 \end{matrix}$$

u is free

choose $u = 1$

$$\vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $\beta = \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

is an eigenbasis for T and we could show through explicit calculation

$$[T]_\beta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$