HOMEWORK 3 ON FINDING INVERSES & DETERMINANTS

§2.4 # 8, 60, 67, 84 // §3.1 # 20, 22, 24, 26, 76 // §3.2 # 8, 66

FROM SPENCE, INSEL & FRIEDBERG'S ELEMENTARY LINEAR ALG. 2nd Ed.

§2.4#8 Suppose $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 5 \\ 1 & 3 & 1 \end{bmatrix}$. Determine if A^{-1} exists and if it does then calculate it. As I proved in notes we can reduce $[A|I]$ to find A^{-1}

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

Continuing,

$$\xrightarrow[r_2 + r_3]{r_1 + 2r_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & 0 & 2 \\ 0 & -1 & 0 & -3 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{r_1 + 3r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 3 & 5 \\ 0 & -1 & 0 & -3 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

Again continuing,

$$\xrightarrow[-r_3]{-r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 3 & 5 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] = [I|A^{-1}]$$

according to my notes, the text etc...

Let's check,

$$\begin{bmatrix} -10 & 3 & 5 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 5 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore \quad \boxed{A^{-1} = \begin{bmatrix} -10 & 3 & 5 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}}$$

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§2.4#60 Given $x_1 + x_2 + x_3 = -5$, $2x_1 + x_2 + x_3 = -3$, $3x_1 + x_3 = 2$. Solve this system by "multiplication by inverse."

In matrix notation,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix} \longleftrightarrow Ax = b \quad \text{with } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$$

Calculate the inverse of the coefficient matrix by our usual algorithm,

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -3 & -2 & -3 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - 3r_2]{r_1 + r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 + r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right]. \text{ Then } Ax = b \Rightarrow x = A^{-1}b = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 2 & -1 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix} = \underline{\begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}}$$

$$\xrightarrow{-r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}\right] \quad \underbrace{}_{A^{-1}}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 3 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

§2.4 # 67 Let $A \in \mathbb{R}^{n \times n}$ such that $A^k = I_n$ for some $k \in \mathbb{N}$. Show A^{-1} exists and find a formula for it.

It's easier to do part (b.) first since once we have a formula for A^{-1} that proves A is invertible.
By definition of A^k we have $A^k = AA^{k-1}$, but this gives us $AA^{k-1} = I$. By a prop. in the notes we also have $A^{k-1}A = I$. Therefore, $A^{-1} = A^{k-1}$ by the definition of inverse.

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§2.4 # 84 Let $A, B, C \in \mathbb{R}^{n \times n}$. Prove the following

(a.) A is similar to A.

(b.) If A is similar to B then B is similar to A.

(c.) If A is similar to B and B is similar to C then A is sim. to C.

We define $A \sim B$ iff $\exists$ invertible $P \in \mathbb{R}^{n \times n}$ such that $B = P^{-1}AP$.
I use $A \sim B$ as shorthand for "A similar to B"

(a.) Notice $A = I^{-1}AI = IAI = A \quad \therefore \quad A \sim A$.

(b.) Assume $A \sim B$ then $\exists P \in \mathbb{R}^{n \times n}$ s.t. $B = P^{-1}AP$
multiply by P on left and P^{-1} on right to obtain,

$$PBP^{-1} = PP^{-1}APP^{-1} = IAI = A.$$

Notice $Q = P^{-1}$ has $Q^{-1} = (P^{-1})^{-1} = P$ thus we can write $PBP^{-1} = Q^{-1}BQ = A \quad \therefore \quad B \sim A$.

(c.) Assume $A \sim B$ and $B \sim C$, then $\exists P, Q$ such that $B = P^{-1}AP$ and $C = Q^{-1}BQ$. Consider that

$$\begin{aligned} C = Q^{-1}BQ &= Q^{-1}(P^{-1}AP)Q \quad (\text{by } A \sim B) \\ &= (PQ)^{-1}A(PQ) \quad \text{by socks-shoes prop. of inverse.} \end{aligned}$$

Thus $A \sim C$ by the similarity transform by PQ.

§3.1 #20 Calculate the determinant of $\begin{bmatrix} 0 & -1 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & -2 & 2 & 3 \\ 0 & 1 & 0 & -2 \end{bmatrix}$ via the co-factor expansion along the 4^{th} row

$$\det\begin{bmatrix} 0 & -1 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & -2 & 2 & 3 \\ 0 & 1 & 0 & -2 \end{bmatrix} = -0\cdot\det\begin{bmatrix} -1 & 0 & 1 \\ 3 & 1 & 4 \\ -2 & 2 & 3 \end{bmatrix} + 1\cdot\det\begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 4 \\ 1 & 2 & 3 \end{bmatrix} - 0\cdot\det\begin{bmatrix} 0 & -1 & 1 \\ -2 & 3 & 4 \\ 1 & -2 & 3 \end{bmatrix} - 2\det\begin{bmatrix} 0 & -1 & 0 \\ -2 & 3 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

$$= \det\begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 4 \\ 1 & 2 & 3 \end{bmatrix} - 2\det\begin{bmatrix} 0 & -1 & 0 \\ -2 & 3 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

$$= -5 - 2(-5)$$

$$= \boxed{5.}$$

§3.1 #22

$$\det\begin{bmatrix} 8 & 0 & 0 \\ -1 & -2 & 0 \\ 4 & 5 & 3 \end{bmatrix} = 8\det\begin{bmatrix} -2 & 0 \\ 5 & 3 \end{bmatrix} = 8(-6-0) = \boxed{-48.}$$

§3.1 #24

$$\det\begin{bmatrix} 7 & 1 & 8 \\ 0 & -3 & 4 \\ 0 & 0 & -2 \end{bmatrix} = -2(-1)^{3+3}\det\begin{bmatrix} 7 & 1 \\ 0 & -3 \end{bmatrix} = -2(-21) = \boxed{42.}$$

§3.1 #26

$$\det\begin{bmatrix} 5 & 1 & 1 \\ 0 & 2 & 0 \\ 6 & -4 & 3 \end{bmatrix} = 2(-1)^{2+2}\det\begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} = 2(15-6) = \boxed{18.}$$

§3.1 #76 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $E = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. Verify $\det(EA) = \det E \det A$

$$EA = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$$

Thus $\det(EA) = (a+kc)d - (b+kd)c = (ad-bc) + k(cd-dc)$ or simply $\det(EA) = ad-bc$. Next note $\det(E) = 1$ and $\det(A) = ad-bc$. It follows $ad-bc = \det(EA) = \det(E)\det(A)$.

§ 3.2 #8 Find determinant by expanding the cofactors along the 3rd Column.

$$\det\begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = 2(-1)^{2+3}\det\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} + (-1)^{3+3}\det\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= 0 - 2$$

$$= \boxed{-2}$$

§ 3.2 #66

$$\begin{aligned} -2x_1 + 3x_2 + x_3 &= -2 \\ 3x_1 + x_2 - x_3 &= 1 \\ -x_1 + 2x_2 + x_3 &= -1 \end{aligned} \longrightarrow \begin{bmatrix} -2 & 3 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

Cramer's Rule says:

$$x_1 = \frac{\det\begin{bmatrix} -2 & 3 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}}{\det\begin{bmatrix} -2 & 3 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}} = \frac{-2(1+2) - 3(1-1) + 1(2+1)}{-2(1+2) - 3(3-1) + 1(6+1)} = \frac{-3}{-5}$$

$$x_2 = \frac{\det\begin{bmatrix} -2 & -2 & 1 \\ 3 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}}{\det\begin{bmatrix} -2 & 3 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}} = \frac{-2(1-1) + 2(3-1) + 1(-3+1)}{-5} = \frac{2}{-5}$$

$$x_3 = \frac{\det\begin{bmatrix} -2 & 3 & -2 \\ 3 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}}{-5} = \frac{-2(-1-2) - 3(-3+1) - 2(6+1)}{-5}$$

$$= \frac{6 + 6 - 14}{-5}$$

$$= \frac{-2}{5}$$

$$-2(1+2) - 3(3-1) + 1(6+$$
$$-2(3) - 3(2) + 7$$
$$-12 + 7$$

$$\boxed{x_1 = 3/5,\quad x_2 = -2/5,\quad x_3 = -2/5}$$

Check answer: $-2\left(\frac{3}{5}\right) + 3\left(\frac{-2}{5}\right) + \frac{2}{5} = \frac{-10}{5} = -2.$