Homework 5: General Sol^{n}, Row, Column & Null Spaces

From Spence, Insel & Friedberg // Elementary Linear Algebra 2nd Ed.

§1.7 # 54 // §4.1 # 20, 28, 82, 90 // §4.2 # 6 // §4.3 # 4

§1.7 # 54 Write vector form of general sol^{n} of $x_1 + 4x_4 = 0$, $x_2 - 2x_4 = 0$

Notice $x_1 = -4x_4$ and $x_2 = 2x_4$ and x_3 is also free thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4x_4 \\ 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

§4.1 # 20 Let $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 3 & -2 \\ -2 & 3 & -1 & 2 \end{bmatrix}$. Is $\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \in \mathrm{Col}(A)$?

In other words, does $Ax = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ have a sol^{n}? Consider

$$\mathrm{rref}\left[\begin{array}{cccc|c} 1 & -2 & -1 & 0 & -1 \\ 0 & 1 & 3 & -2 & 3 \\ -2 & 3 & -1 & 2 & -1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 5 & -4 & 5 \\ 0 & 1 & 3 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Yes, $5\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \in \mathrm{Col}(A)$.

(I used the CCP to see the linear combo. here)

§4.1 # 28 Find (basis) generating set for Null(A) given that $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow[r_3 - 2r_2]{r_1 + 2r_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has sol^{n} with $x + 2z = 0$ & $-y + z = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \therefore \ \mathrm{Null}(A) = \mathrm{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

$\boxed{\S 4.1 \# 82}$ Show $\left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^{2\times 1} \,\middle|\, 2u_1^2 + 3u_2^2 = 12 \right\} = W$ is <u>not</u> a subspace of $\mathbb{R}^{2\times 1}$.

Observe that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{2\times 1}$ has $2(0)^2 + 3(0)^2 = 0 \neq 12$ $\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin W$ hence $W \not\leq \mathbb{R}^{2\times 1}$. There are many other correct answers here. You just have to find <u>one</u> way W <u>fails</u> to be a subspace.

$\boxed{\S 4.1 \# 90}$ Show $W = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^{2\times 1} \,\middle|\, 5u_1 + 4u_2 = 0 \right\} \leq \mathbb{R}^{2\times 1}$

Note $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has $5(0) + 4(0) = 0$ $\therefore \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$. Suppose $\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \in W$ then by def$^{\underline{n}}$ of W we have $5x + 4y = 0$ and $5a + 4b = 0$. Note that

$$5(x+a) + 4(y+b) = (5x+4y) + (5a+4b) = 0 + 0 = 0$$

hence $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix} \in W$ so W is closed under addition. Next consider $\begin{bmatrix} x \\ y \end{bmatrix} \in W$ and $c \in \mathbb{R}$,

$$\begin{aligned} 5x + 4y = 0 \quad &\Rightarrow 5cx + 4cy = 0 \\ &\Rightarrow [cx, cy]^T \in W \\ &\Rightarrow c[x, y]^T \in W \\ &\Rightarrow W \text{ closed under scalar multiplication.} \end{aligned}$$

<u>Thus, by the subspace test, $W \leq \mathbb{R}^{2\times 1}$.</u>

§4.2 #6 Find a basis for the column space and null space of the matrix $A = \begin{bmatrix} 1 & 1 & -1 & -2 \\ -1 & -2 & 1 & 3 \\ 2 & 3 & 1 & 4 \end{bmatrix}$

We can calculate,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \beta = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$ is basis for Col (A).

$$\text{Null}(A) = \{ x = [x_1, x_2, x_3, x_4]^T \mid Ax = 0 \}$$

$$x = [x_1, x_2, x_3, x_4]^T \in \text{Null}(A) \Rightarrow \begin{aligned} x_1 &= -2x_4 \\ x_2 &= x_4 \\ x_3 &= -3x_4 \end{aligned}$$

$$\Rightarrow x = [x_1, x_2, x_3, x_4]^T$$

$$\Rightarrow x = x_4 [-2, 1, -3, 1]^T$$

$$\Rightarrow \text{Null}(A) = \text{span} \Big\{ \underbrace{\begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix}}_{\text{basis for Null}(A)} \Big\}$$

§4.3 #4 Given that A is a matrix with

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

determine the dimensions of column, Null, row space of A and also dim (Null (A^T))

(a.) $\boxed{\dim(\text{Col}(A)) = 3}$ since there are 3 pivot columns

(b.) $\boxed{\dim(\text{Null}(A)) = 2}$ since the sol$^{\underline{n}}$ of $Ax = 0$ has <u>two</u> free variables. In fact we could even find the basis for Null(A) despite our ignorance of A.

(c.) $\boxed{\dim(\text{Row}(A)) = 3}$ since there are 3 nonzero rows in rref(A) and Row$(A) = \text{span}\{[1\ 0\ 0\ {-4}\ 2], [0\ 1\ 0\ 2\ {-1}], [0\ 0\ 1\ {-3}\ 1]\}$.

(d.) $\text{Null}(A^T) = \{ x \in \mathbb{R}^{4\times 1} \mid A^T x = 0 \}$. There are many ways to argue this. I'll use rank(A) + nullity$(A) = n$ applied to A^T:

$$\text{rank}(A^T) + \text{nullity}(A^T) = 4.$$

$$\text{rank}(A^T) = \dim(\text{row}(A)) \text{ since } \text{Col}(A^T) = \text{Row}(A) \Rightarrow \boxed{\text{nullity}(A^T) = 1}$$