HOMEWORK 6: LINEAR TRANSFORMATIONS & ISOMORPHISMS

§2.7 # 4, 28, 60, 72 // §2.8 # 24, 38, 88 // §4.2 #12 // §7.2 # 4 // §7.3 # 66 // §7.4 # 46a-b

FROM SPENCE INSEL & FRIEDBERG's *Elementary Linear Algebra* 2nd Ed.

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§2.7 #4 Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 0 & -2 \end{bmatrix}$. Give the domain and codomain of the linear transformation induced from A

We define $L_A(x) = Ax$, this makes L_A the linear transformation induced from A. To figure out the dimensions just think about what x must be for Ax to be a well-defined matrix product

oops the text put A^T here See below

Since A is $2 \times 3 \Rightarrow x$ is 3×1.

But then Ax is $(2\times3)(3\times1) \longrightarrow (2\times1)$. Hence

$$L_A : \underset{\text{domain}}{\mathbb{R}^{3\times1}} \longrightarrow \underset{\text{codomain.}}{\mathbb{R}^{2\times1}}$$

Remark: I solved wrong problem here, look below for A^T

§2.7 #28 Find the standard matrix of T given that $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_2 \\ 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_2 \\ 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 0 & 3 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore \boxed{[T] = \begin{bmatrix} 0 & 3 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}} \leftarrow \text{standard matrix of } T.$$

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§2.7#4, actually was for A^T

same logic applies A^T is $3 \times 2 \Rightarrow A^Tx$ is well-def. if x is (2×1).

$$\boxed{L_{A^T} : \underset{\text{(domain)}}{\mathbb{R}^{2\times1}} \longrightarrow \underset{\text{(codomain)}}{\mathbb{R}^{3\times1}}}$$

§ 2.7 # 28 Given $T =: \mathbb{R}^{2\times1} \longrightarrow \mathbb{R}^{3\times1}$ is defined by

$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}3x_2\\2x_1-x_2\\x_1+x_2\end{bmatrix}$ find standard matrix of T

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}0 & 3\\2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

§ 2.7 # 60 Suppose $T : \mathbb{R}^{2\times1} \longrightarrow \mathbb{R}^{2\times1}$ is linear and

$T\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = \begin{bmatrix}-4\\6\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\3\end{bmatrix}\right) = \begin{bmatrix}9\\-6\end{bmatrix}$.

Calculate $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ (and explain how you did it)

$$T(e_1) = \underbrace{\tfrac{1}{2}T(2e_1)}_{\text{used homogeneity of } T.} = \tfrac{1}{2}T\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix}-4\\6\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix}.$$

$$T(e_2) = \overbrace{\tfrac{1}{3}T(3e_2)} = \tfrac{1}{3}T\left(\begin{bmatrix}0\\3\end{bmatrix}\right) = \tfrac{1}{3}\begin{bmatrix}9\\-6\end{bmatrix} = \begin{bmatrix}3\\-2\end{bmatrix}$$

$$\therefore \quad [T] = \left[T(e_1) \,\middle|\, T(e_2)\right] = \begin{bmatrix}-2 & 3\\3 & -2\end{bmatrix}$$

$$\therefore \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}-2 & 3\\3 & -2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \boxed{\begin{bmatrix}4\\-1\end{bmatrix} = T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)}$$

Remark: there are other sol$^{\underline{n}}$/s.

§27 # 72 Is $T : \mathbb{R}^{2\times1} \longrightarrow \mathbb{R}^{2\times1}$ with $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}2x_1\\x_2^2\end{bmatrix}$

is T linear? Explain why or why not.

$$T\left(c\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}0\\c\end{bmatrix}\right) = \begin{bmatrix}0\\c^2\end{bmatrix} = c^2\begin{bmatrix}0\\1\end{bmatrix} = c^2\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$\therefore \quad T\left(c\begin{bmatrix}0\\1\end{bmatrix}\right) \neq c\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

(counter-ex. against homogeneity.)

§2.8 # 24 | Find standard matrix of T and use it to determine whether T is one-one, $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 + x_2 \end{bmatrix}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \therefore [T] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$\underline{\text{Null}([T]) = ?}$ We know T is 1-1 iff $\text{Null}([T]) = \{0\}$. Thus, we need to calculate $\text{Null}([T])$. Notice $\det([T]) = -1 \neq 0 \therefore [T]^{-1}$ exists. Thus if $[T]x = 0 \Rightarrow [T]^{-1}[T]x = [T]^{-1}x = 0 \Rightarrow x = 0$.

Hence, $\text{null}([T]) = \{0\} \Rightarrow \boxed{T \text{ is } 1\text{-}1}$

Remark: in the case $T : \mathbb{R}^{n\times 1} \longrightarrow \mathbb{R}^{m\times 1}$ we have $\text{Kernel}(T) = \text{Null}([T])$. My apologies if I inadvertently confuse the notations from time to time, various books use different notation on this topic. The proof that T is 1-1 iff $\text{Kernel}(T) = \{0\}$ is not too tricky. Let me argue that here since I think it's currently missing from my notes.

Th$^{\underline{m}}$/ Let $T : V \to W$ be a linear operator, T is 1-1 iff $\text{Ker}(T) = \{0\}$

$\Rightarrow$| Assume T is 1-1. This means $T(x) = T(y) \Rightarrow x = y \;\; \forall x, y \in V$. Let $x \in \text{Ker}(T)$ and note $T(0) = 0$ thus $T(x) = T(0) = 0$ and by 1-1 prop. of T we find $x = 0$ hence $\text{Ker}(T) \subseteq \{0\}$ and clearly $\{0\} \subseteq \text{Ker}(T)$ since $T(0) = 0 \therefore \text{Ker}(T) = \{0\}$.

$\Leftarrow$| Assume $\text{Ker}(T) = \{0\}$. Suppose $x, y \in V$ and $T(x) = T(y) \Rightarrow T(x) - T(y) = 0 \Rightarrow T(x - y) = 0$ Thus $(x - y) \in \text{Ker}(T)$. But $\text{Ker}(T) = \{0\} \Rightarrow x - y = 0 \therefore x = y$. Hence T is 1-1.

(FEEL FREE TO USE THIS THEOREM ON TEST ETC...)

§2.8#38 Find the standard matrix of $T:\mathbb{R}^{4\times1}\to\mathbb{R}^{4\times1}$ defined below and use $[T]$ to determine whether T is onto.

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right)=\begin{bmatrix}x_1-x_2+2x_3\\-2x_1+x_2-7x_3\\x_1-x_2+2x_3\\-x_1+2x_2+x_3\end{bmatrix}$$

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right)=\underbrace{\begin{bmatrix}1&-1&2&0\\-2&1&-7&0\\1&-1&2&0\\-1&2&1&0\end{bmatrix}}_{[T]\text{ the standard matrix of }T.}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$\text{rref}\,[T]=\begin{bmatrix}1&0&5&0\\0&1&3&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\Rightarrow \text{range}(T)=\text{span}\left\{\begin{bmatrix}1\\-2\\1\\-1\end{bmatrix},\begin{bmatrix}-1\\1\\-1\\2\end{bmatrix}\right\}$$

$\Rightarrow \text{rank}(T)=2\neq4 \;\therefore$ $\underline{T\text{ is not onto}}$.

§2.8#88 Define $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x-y+2z\\-x+2y-3z\\2x+z\end{bmatrix}$ find T^{-1}

Notice that $T^{-1}(v)=[T]^{-1}v$ since $T(T^{-1}(v))=[T]T^{-1}(v)=v$ for all v $\Rightarrow [T][T^{-1}]=I \;\therefore\; [T]^{-1}=[T^{-1}]$. So we can find inverse transformation by inverting $[T]$,

$$\left[[T]\,|\,I\right]=\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\-1&2&-3&0&1&0\\2&0&1&0&0&1\end{array}\right]\longrightarrow \text{rref}\left([T]|I\right)=\left[\begin{array}{ccc|c}1&0&0&\\0&1&0&\underbrace{\begin{matrix}2&-1&1\\5&3&-1\\4&2&-1\end{matrix}}_{[T]^{-1}}\\0&0&1&\end{array}\right]$$

$$\boxed{T^{-1}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{bmatrix}2&-1&1\\5&3&-1\\4&2&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2x-y+z\\5x+3y-z\\4x+2y-z\end{bmatrix}}.$$

Check Answer: $T\left(T^{-1}\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=T\begin{pmatrix}2x-y+z\\5x+3y-z\\4x+2y-z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}\leftarrow\left(\text{a short calculation}\right)$

§4.2 #12 Find (a) basis for range(T), (b.) basis for Null(T) if Null(T) $\neq \{0\}$, given that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_3 + x_4 \\ x_1 + 3x_3 + 2x_4 \\ -x_1 + x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(a.) $$\text{rref}\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ -1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \uparrow & & \uparrow & \uparrow \end{matrix}$$ pivot columns.

$$\Rightarrow \boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\} \text{ BASIS FOR range(T).}}$$

(b.) Given rref([T]) above we can solve $[T]x = 0$ for $x = [x_1, x_2, x_3, x_4]^T$ with ease, notice x_2 is free while,

$$\left.\begin{matrix} x_1 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{matrix}\right\} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\text{BASIS OF Null([T]) is simply } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$$

Remark: I know these are bases because of theory we discussed in lecture. The column space is spanned by the LI pivot cols. and null space is spanned by vectors appearing in the vector form of the sol$^{\underline{n}}$ to $[T]x = 0$.

§ 7.2 #4 Determine if the linear transformation $T : P_2 \longrightarrow \mathbb{R}^{2\times 1}$ defined by $T(f(x)) = \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix}$ is 1-1.

It suffices to show $\operatorname{Ker}(T) = \{0\}$.

$$\begin{aligned}\operatorname{Ker}(T) &= \{ f(x) \in P_2 \mid T(f(x)) = 0 \} \\ &= \{ ax^2+bx+c \in P_2 \mid [a+b+c,\ 2a+b] = 0 \} \\ &= \{ ax^2+bx+c \in P_2 \mid a+b+c = 0,\ 2a+b = 0 \}\end{aligned}$$

Notice that we have 2-eq$^{\underline{n}}$s but 3-unknowns a, b, c thus there will be ∞ly many sol$^{\underline{n}}$s (since this system is consistent). I'll solve with a as the free variable,

$$\begin{aligned} a &= a \\ b &= -2a \\ c &= -a-b = -a+2a = a \end{aligned}$$

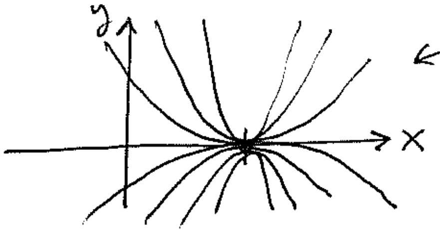
Thus, continuing the $\operatorname{Ker}(T)$ calculation,

$$\begin{aligned}\operatorname{Ker}(T) &= \{ ax^2+bx+c \in P_2 \mid b = -2a,\ c = a \} \\ &= \{ a(x^2-2x+1) \mid a \in \mathbb{R} \} \\ &\neq \{0\}.\end{aligned}$$

Thus T is not 1-1.

$$\left(\quad T(a(x^2-2x+1)) = T(a(x-1)^2) = 0 \right)$$

Graphically,



$\leftarrow$ T maps all these fncts to zero.

§ 7.3 #66 Given $T: V \longrightarrow W$ is an isomorphism,

a.) Prove if $\beta = \{V_1, V_2, .., V_n\}$ is LI then $T(\beta) = \{T(V_1), .., T(V_n)\}$ LI

b.) Prove if β is basis for V then $T(\beta)$ is basis for W.

c.) Prove if V is finite dimensional then W is also and $\dim(V) = \dim(W)$

a.) Assume $\beta = \{V_1, V_2, V_3, \dots, V_n\} \subseteq V$ is LI. Consider $\{T(V_1), T(V_2), \dots, T(V_n)\} \subseteq W$ (we know $T(V_i) \in W$ for each $i = 1, 2, .., n$ since $T: V \longrightarrow W$). Suppose

$$c_1 T(V_1) + c_2 T(V_2) + \cdots + c_n T(V_n) = 0$$

Since T is an isomorphism we have T^{-1} exists and it is also a linear transformation. Thus,

$$T^{-1}(c_1 T(V_1) + c_2 T(V_2) + \cdots + c_n T(V_n)) = T^{-1}(0)$$

$$\Rightarrow c_1 T^{-1}(T(V_1)) + c_2 T^{-1}(T(V_2)) + \cdots + c_n T^{-1}(T(V_n)) = 0$$

$$\Rightarrow c_1 V_1 + c_2 V_2 + \cdots + c_n V_n = 0$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0 \text{ by LI of } \beta.$$

Hence $T(\beta)$ is LI subset of W. //

b.) Assume β is a basis for V. Thus $\beta = \{V_1, V_2, .., V_n\}$ spans V and β is LI. By part (a.) we have $T(\beta)$ is LI. Suppose $w \in W$ note $T^{-1}(w) \in V = \operatorname{span}\beta$ thus $\exists\, c_1, c_2, .., c_n$ s.t.

$$T^{-1}(w) = c_1 V_1 + c_2 V_2 + \cdots + c_n V_n$$

$$\Rightarrow T(T^{-1}(w)) = w = T(c_1 V_1 + c_2 V_2 + \cdots + c_n V_n)$$

$$\Rightarrow w = c_1 T(V_1) + c_2 T(V_2) + \cdots + c_n T(V_n).$$

Thus $W \subseteq \operatorname{span}(T(\beta))$. Notice ~~$T(\beta)$~~ $\operatorname{span}(T(\beta)) \subseteq W$ is almost obvious. Thus we find $\operatorname{span} T(\beta) = W$. Therefore $T(\beta)$ is a basis for W since it's LI and it spans. //

§ 7.3#66 Continued

(c.) If V is finite dimensional then $\exists\ \beta = \{v_1, v_2, \dots, v_n\}$ which is a basis for V. By part (b.) $T(\beta) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis for W. Recall that T is an isomorphism & hence it is a bijection a.k.a. a 1-1 correspondance thus $T(\beta)$ has same # of objects as β.

$\therefore$ $\dim(W) = n$. //

§7.4#48a-b Let $B \in \mathbb{R}^{n\times n}$ and $T: \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ be defined by $T(A) = BA$. Prove that
(a.) T is linear (b.) T is invertible iff B^{-1} exists.

a.)
$$\begin{aligned} T(A_1 + A_2) &= B(A_1 + A_2) \\ &= BA_1 + BA_2 \\ &= T(A_1) + T(A_2) \quad \forall A_1, A_2 \in \mathbb{R}^{n\times n} \end{aligned}$$

$\therefore$ T is additive.

$$\begin{aligned} T(cA) &= B(cA) \\ &= c(BA) \\ &= c\,T(A) \quad \forall A \in \mathbb{R}^{n\times n} \text{ and } c \in \mathbb{R} \end{aligned}$$

$\therefore$ T is homogeneous.

Thus, T is linear.

§ 7.4 # 46 b / T^{-1} exists $\Leftrightarrow$ B^{-1} exists.

$\Leftarrow$ | If B^{-1} exists then define $T^{-1}(X) = B^{-1}X \quad \forall X \in \mathbb{R}^{n\times n}$.

Observe $T(T^{-1}(x)) = T(B^{-1}(X) = B(B^{-1}x) = X \quad \forall x \in \mathbb{R}'$

and $T^{-1}(T(X)) = T^{-1}(BX) = B^{-1}BX = X$. Thus

T^{-1} (as we defined it) is truly the inverse of T.//

$\Rightarrow$ | Assume T^{-1} exists. We have that

① $T^{-1}(T(X)) = X \quad \forall X \in \mathbb{R}^{n\times n}$

② $T(T^{-1}(x)) = X \quad \forall X \in \mathbb{R}^{n\times n}$

We also know $T(x) = Bx$ by def$^{\underline{n}}$ of T.

Hence, $\forall X \in \mathbb{R}^{n\times n}$ we have
$T^{-1}(BX) = X$ & $B\,T^{-1}(X) = X$

Choose $X = I$ to make ② interesting,

$$B\,T^{-1}(I) = I$$

$$\Rightarrow \det(B)\det(T^{-1}(I)) = \det(I) = 1$$

$$\therefore \det(B) \neq 0 \Rightarrow \underline{B^{-1} \text{ exists}}.//$$

To summarize, T^{-1} exists iff B^{-1} exists given that $T(A) = BA$ defines $T: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$.