Homework 9: Eigenvalues, Eigenvectors...

§ 5.2 # 14, 16, 38, 34 // § 7.5 # 6, 12, 20, 47, 72

§ 5.2 # 14 Find e-values & eigenbasis for each e-space of $A = \begin{bmatrix} 8 & 2 \\ -12 & -2 \end{bmatrix}$

$$\begin{aligned}
\det(A - \lambda I) &= \det\begin{pmatrix} 8-\lambda & 2 \\ -12 & -2-\lambda \end{pmatrix} \\
&= (8-\lambda)(-2-\lambda) + 24 \\
&= \lambda^2 + 2\lambda - 8\lambda - 16 + 24 \\
&= \lambda^2 - 6\lambda + 8 \\
&= (\lambda - 2)(\lambda - 4) \\
&= 0 \implies \underline{\lambda_1 = 2 \ \& \ \lambda_2 = 4}.
\end{aligned}$$

$\lambda_1 = 2$)

$$(A - 2I)u_1 = \begin{bmatrix} 6 & 2 \\ -12 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{array}{c} 6u + 2v = 0 \\ v = -3u \end{array}$$

$\longrightarrow$ choose $u_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

basis for $W_{\lambda=2}$

$\lambda = 4$)

$$(A - 4I)u_2 = \begin{bmatrix} 4 & 2 \\ -12 & -6 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{array}{c} 4u + 2v = 0 \\ v = -2u \end{array}$$

$\longrightarrow$ choose $u_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

basis for $W_{\lambda=4}$

Defn/ $W_{\lambda=\lambda_0} = \{ x \in \mathbb{R}^{n\times 1} \mid Ax = \lambda_0 x \}$
the λ_0-eigenspace associated to the matrix A.

Note:
We can choose any nonzero e-vector to give a basis for a 1-dim'l e-space. Of course generally $W_{\lambda=\lambda_0}$ could be two or three or higher dim-l, in which case we'd have more free variables in the eqn $(A - \lambda_0 I)u = 0$.

$\boxed{\S 5.2\ \#16}$ Same as #14, but $A = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} -2-\lambda & 0 \\ 3 & -1-\lambda \end{pmatrix} = (\lambda+1)(\lambda+2) = 0 \therefore \boxed{\lambda_1 = -1,\ \lambda_2 = -2}$$

$$(A + I)u_1 = \begin{bmatrix} -1 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} u = 0 \\ v \text{ free} \end{matrix} \longrightarrow \boxed{u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\text{with } \lambda_1 = -1)}$$

$$(A + 2I)u_2 = \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} 3u + v = 0 \\ v = -3u \end{matrix} \longrightarrow u_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad (\text{with } \lambda_2 = -$$

$\boxed{\S 5.2\ \#34}$ Find e-values and sample e-vectors for the linear operator $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 6x_1 - x_2 \\ 6x_1 + x_2 \end{bmatrix}$

Note that $[T] = \begin{bmatrix} 6 & -1 \\ 6 & 1 \end{bmatrix}$ thus consider,

$$\begin{aligned} \det([T] - \lambda I) = \det\begin{pmatrix} 6-\lambda & -1 \\ 6 & 1-\lambda \end{pmatrix} &= (\lambda-6)(\lambda-1) + 6 \\ &= \lambda^2 - 7\lambda + 6 + 6 \\ &= \lambda^2 - 7\lambda + 12 \\ &= (\lambda - 3)(\lambda - 4) \to \boxed{\lambda_1 = 3,\ \lambda_2 = 4} \end{aligned}$$

$\underline{\lambda_1 = 3}$ $\begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} 3u - v = 0 \\ v = 3u \end{matrix} \xrightarrow[\text{choose.}]{} \boxed{u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$

$\underline{\lambda_2 = 4}$ $\begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} 2u - v = 0 \\ v = 2u \end{matrix} \longrightarrow \boxed{u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

We have $([T] - 3I)u_1 = 0$ and $([T] - 4I)u_2 = 0$ thus it follows that $T(u_1) = 3u_1$ and $T(u_2) = 4u_2$ which proves that u_1 is e-vector with e-value $\lambda_1 = 3$ and u_2 is e-vector with e-value $\lambda_2 = 4$.

$\boxed{\text{§5.4 \# 38 | Same as \#34 but, } T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -6x_1 - 5x_2 + 5x_3 \\ -x_2 \\ -10x_1 - 10x_2 + 9x_3 \end{bmatrix}}$

Observe $[T] = \begin{bmatrix} -6 & -5 & 5 \\ 0 & -1 & 0 \\ -10 & -10 & 9 \end{bmatrix}$

$$\det([T] - \lambda I) = \det\begin{bmatrix} -6-\lambda & -5 & 5 \\ 0 & -1-\lambda & 0 \\ -10 & -10 & 9-\lambda \end{bmatrix}$$

$$= (-1-\lambda)\det\begin{bmatrix} -6-\lambda & 5 \\ -10 & 9-\lambda \end{bmatrix}$$

$$= -(\lambda+1)\left[(\lambda-9)(\lambda+6) + 50\right]$$

$$= -(\lambda+1)\left[\lambda^2 - 3\lambda - 54 + 50\right]$$

$$= -(\lambda+1)(\lambda^2 - 3\lambda - 4)$$

$$= -(\lambda+1)^2(\lambda-4) \quad \therefore \boxed{\lambda_1 = -1 = \lambda_2 \ \ \& \ \ \lambda_3 = 4}$$

$\underline{\lambda_1 = \lambda_2 = -1}$ | $u_{1,2} = [u, v, w]^T$ as usual, solve $([T] + I)u_{1,2} = 0$

$$\begin{bmatrix} -5 & -5 & 5 \\ 0 & 0 & 0 \\ -10 & -10 & 10 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow -u - v + w = 0$$

$\longrightarrow w = u + v$

think of u, v as free and w as dependent

<u>Many</u> choices possible, I choose $u = 0,\ v = 1 \rightarrow u_2$; $u = 1,\ v = 0 \rightarrow u_1$

$$\boxed{u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

$(W_{\lambda=-1} = \text{span}\{u_1, u_2\})$

$\underline{\lambda_3 = 4}$ | Solve $([T] - 4I)u_3 = 0$

$$\begin{bmatrix} -10 & -5 & 5 \\ 0 & -5 & 0 \\ -10 & -10 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \underline{v = 0}.$$

$$-10u + 5w = 0$$

$$w = -2u \quad \therefore \quad \boxed{u_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}$$

$(W_{\lambda=4} = \text{span}\{u_3\})$

§7.5 #6 We define $\langle f, g \rangle = \int_1^2 f(t)\,g(t)\,dt$ calculate $\langle t^2, \frac{1}{t} \rangle$.

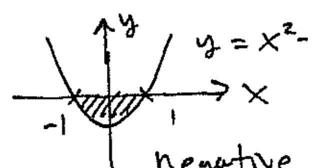
$$\langle t^2, \tfrac{1}{t} \rangle = \int_1^2 t^2 \frac{1}{t}\,dt = \int_1^2 t\,dt = \frac{t^2}{2}\Big|_1^2 = \frac{4}{2} - \frac{1}{2} = \boxed{\frac{3}{2}.}$$

§7.5 #12 $\langle A, B \rangle = \operatorname{trace}(AB^T)$ calculate $\langle A, B \rangle$ given $A = \begin{bmatrix} 0 & 5 \\ -2 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$\begin{aligned} \langle A, B \rangle &= \operatorname{trace}\left(\begin{bmatrix} 0 & 5 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) \\ &= \operatorname{trace}\left(\begin{bmatrix} 15 & 20 \\ -2 & -4 \end{bmatrix} \right) \\ &= 15 - 4 = \boxed{11.} \end{aligned}$$

§7.5 #20 $\langle f(x), g(x) \rangle = \int_{-1}^1 f(t)\,g(t)\,dt$
Let $f(x) = x+1$ and $g(x) = x-1$ calculate $\langle f(x), g(x) \rangle$.

$$\begin{aligned} \langle f(x), g(x) \rangle &= \int_{-1}^1 (x+1)(x-1)\,dx \\ &= \int_{-1}^1 (x^2-1)\,dx \\ &= \left(\frac{x^3}{3} - x \right|_{-1}^1 \\ &= \left(\frac{1}{3} - 1 \right) - \left(\frac{-1}{3} + 1 \right) \\ &= \frac{-2}{3} - \frac{2}{3} \\ &= \boxed{\frac{-4}{3}} \end{aligned}$$



negative area ✓

ok.

(just checking)

§7.5 #47 Let V be a finite dimensional vector space with basis β. Define $\langle u,v\rangle = [u]_\beta \cdot [v]_\beta$ $\forall u,v \in V$. Prove $\langle\, ,\, \rangle$ defines an inner product on V

Observe that since $[u]_\beta \cdot [v]_\beta$ is a dot-product we know that symmetry, additivity & homogeneity automatically follow:

$$\langle u,v\rangle = [u]_\beta \cdot [v]_\beta = [v]_\beta \cdot [u]_\beta = \langle v,u\rangle$$

$$\begin{aligned}\langle u+v, w\rangle &= [u+v]_\beta \cdot [w]_\beta \\ &= ([u]_\beta + [v]_\beta)\cdot [w]_\beta \\ &= [u]_\beta \cdot [w]_\beta + [v]_\beta \cdot [w]_\beta \\ &= \langle u,w\rangle + \langle v,w\rangle\end{aligned}$$

property of coordinate mapping.

$$\begin{aligned}\langle cu, v\rangle &= [cu]_\beta \cdot [v]_\beta \\ &= c\,[u]_\beta \cdot [v]_\beta \\ &= c\,\langle u,v\rangle .\end{aligned}$$

prop. of coord. map.

Finally $\Phi_\beta(v) = [v]_\beta$ is an <u>isomorphism</u> thus $\Phi_\beta(v) = 0$ iff $v = 0$ hence

$$\langle v,v\rangle = [v]_\beta \cdot [v]_\beta = 0 \Rightarrow [v]_\beta = 0$$
$$\Rightarrow v = 0$$

Also $[v]_\beta \cdot [v]_\beta \geq 0$ since this is a dot-product hence $\langle v,v\rangle \geq 0$. Thus $\langle\, ,\, \rangle$ is inner product.

§ 7.5 # 72) Prove that if $A^T = A$ and $B^T = -B$ then A & B are orthogonal w.r.t. Frobenius inner product for matrices ($\langle A, B\rangle = \text{Trace}(AB^T)$)

Observe,

$$\langle A,B\rangle = \text{Trace}(AB^T) = \text{Trace}(B^T A) \quad : \text{ cyclic prop. for trace.}$$
$$= \text{Trace}(-B A^T) \quad : \; B^T = -B, \; A^T = A$$
$$= -\text{Trace}(BA^T) \quad : \text{ trace linear.}$$
$$= -\langle B, A\rangle \quad : \text{ def}^{n} \text{ of } \langle , \rangle$$
$$= -\langle A, B\rangle \quad : \langle , \rangle \text{ is symmetric}$$

$\therefore \quad 2\langle A,B\rangle = 0$

$\therefore \quad \langle A,B\rangle = 0$ hence A & B are orthogonal //