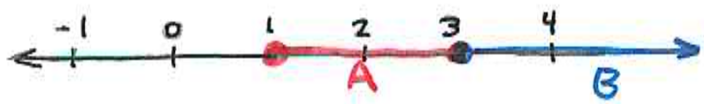
Please print this out and write your solutions on this document. 30pts to earn here. Thanks!

Problem 1: (2pt) Let $A = \{x \in \mathbb{R} \mid |x-2| \leq 1\}$ and $B = \{x \in \mathbb{R} \mid x \geq 3\}$.

(a.) picture A and B on a number line,



Remark: A is collection of points within 1-unit of $x=2$ since $\text{distance}(x,2) = |x-2|$.

(b.) express A and B in interval notation,

Remark: $A: |x-2| \leq 1 \Leftrightarrow -1 \leq x-2 \leq 1 \Leftrightarrow 1 \leq x \leq 3 \Leftrightarrow x \in [1,3]$

$\boxed{B = [3, \infty)}$ so, $\boxed{A = [1,3]}$

(c.) express $A \cup B$ in interval notation,

$\boxed{A \cup B = [1, \infty)}$

(d.) find and express $A \cap B$ in set-builder notation.

$A \cap B = \{3\} = \boxed{\{x \mid x = 3\}}$

perfectly fine answer here, I'll give full credit.

Problem 2: (3pt) Assume $x, y > 0$ and use laws of algebra to determine A, B as indicated below:

(a.) $\dfrac{6x^A}{y^B} = \dfrac{6xy^{-2}}{(x^2y)^3\sqrt{x}} = \dfrac{6xy^{-2}}{(x^2)^3y^3x^{1/2}} = \dfrac{6xy^{-2}}{x^6y^3x^{1/2}} = \dfrac{6x^{1-\frac{1}{2}-6}}{y^{3+2}} = \dfrac{6x^{-11/2}}{y^5}$

We find $\dfrac{6x^A}{y^B} = \dfrac{6x^{-11/2}}{y^5} \Rightarrow \boxed{A = \frac{-11}{2} \text{ and } B = 5}$

(b.) $x^Ay^B = \sqrt{\dfrac{x\sqrt{y}}{x^{-2}y^3}} = \left(\dfrac{x\,y^{1/2}}{x^{-2}y^3}\right)^{1/2} = (x^3y^{-5/2})^{1/2} = x^{3/2}y^{-5/4}$

$\Rightarrow \boxed{A = 3/2 \text{ and } B = -5/4}$

(c.) $x^Ay^B = \left(\sqrt[5]{x^3y^2}\sqrt[3]{x^6y^9}\right)^2 = \left((x^3y^2)^{1/5}(x^6y^9)^{1/3}\right)^2$

$= (x^{3/5}y^{2/5}x^{6/3}y^{9/3})^2$

$= (x^{13/5}y^{17/5})^2$

$= x^{26/5}y^{34/5}$

$\boxed{A = \frac{26}{5}}$

$\boxed{B = \frac{34}{5}}$

D

Problem 3: (2pt) Find the domain of each expression. Please write your answer in interval notation.

(a.) $4x^2 - 9x + 3$

$\boxed{D = (-\infty, \infty)}$ since this is a polynomial whose formula has no restrictions for $x \in \mathbb{R}$.

(b.) $\sqrt{2x+7}$

Need $2x + 7 \geq 0 \Rightarrow x \geq -7/2 \therefore \boxed{D = [-7/2, \infty)}$

(c.) $\dfrac{5x}{x^2+4x+5}$

Observe $x^2 + 4x + 5 = (x+2)^2 + 1$ hence no value of x makes the denominator zero. It follows $\boxed{D = (-\infty, \infty)}$

(d.) $\dfrac{6x+3}{x^2+5x+4}$

Observe $x^2 + 5x + 4 = (x+1)(x+4)$ so we must avoid $x = -1$ and $x = -4$. $D = \{x \mid x \neq -1 \text{ and } x \neq -4\}$ which is $\boxed{D = (-\infty, -4) \cup (-4, -1) \cup (-1, \infty)}$ (interval notation)

Problem 4: (2pt) Peform the addition and simplify the resulting expression.

(a.) $\dfrac{3x-2}{x+1} + 2 = \dfrac{3x-2}{x+1} + 2\left(\dfrac{x+1}{x+1}\right)$

$= \dfrac{3x-2+2(x+1)}{x+1} \quad \left(= \underbrace{\dfrac{3x-2+2x+2}{x+1}}_{\text{Some people might need this step to see the next}}\right)$

$= \boxed{\dfrac{5x}{x+1}}$

(b.) $1 + \dfrac{1}{1+\frac{1}{1+x}} = 1 + \left(\dfrac{1+x}{1+x}\right)\left[\dfrac{1}{1+\frac{1}{1+x}}\right]$

$= 1 + \dfrac{1+x}{(1+x)\left[1+\frac{1}{1+x}\right]}$

$= 1 + \dfrac{1+x}{1+x+\frac{1+x}{1+x}}$

$= 1 + \dfrac{1+x}{1+x+1}$

$= 1 + \dfrac{1+x}{2+x} = \dfrac{2+x+1+x}{2+x} = \boxed{\dfrac{2x+3}{2+x}}$

Problem 5: (2pt) The **standard form** of a polynomial is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$ and $a_n, \cdots, a_1, a_0$ are constants. Multiply the following polynomials and collect like power terms to give your answer in standard form:

(a.) $(x+1)(2x^2 - x + 1)$ $= x(2x^2 - x + 1) + 1(2x^2 - x + 1)$

$= 2x^3 - x^2 + x + 2x^2 - x + 1$

$= \boxed{2x^3 + x^2 + 1}$

(b.) $x^5 + (2x+1)^3$ $= x^5 + (2x+1)(2x+1)(2x+1)$

$= x^5 + (2x+1)[4x^2 + 4x + 1]$

$= x^5 + 2x(4x^2 + 4x + 1) + 4x^2 + 4x + 1$

$= x^5 + 8x^3 + 8x^2 + 2x + 4x^2 + 4x + 1$

$= \boxed{x^5 + 8x^3 + 12x^2 + 6x + 1}$

low-tech brute force solution

$= x^5 + 1(2x)^3 + 3(2x)^2(1)^1 + 3(2x)^1(1)^2 + 1 \cdot 1^3$ binomial Th^m method

1

1 1

1 2 1

1 3 3 1

Problem 6: (6pt) Factor the following polynomials completely over $\mathbb{R}$[1],

(a.) $30x^3 + 15x^4$,

$30x^3 + 15x^4 = \boxed{15x^3(2 + x)}$

(b.) $x^2 - 14x + 48$,

$x^2 - 14x + 48 = \boxed{(x-6)(x-8)}$

(c.) $2x^2 + 7x - 4$,

$2x^2 + 7x - 4 = \boxed{(2x - 1)(x + 4)}$

[1]to factor a polynomial over a set of numbers indicates the kind of coefficients you may use. For example, $x^2 + 1$ is completely factored over $\mathbb{R}$, but $x^2 + 1 = (x+i)(x-i)$ over $\mathbb{C}$.

$$8x^2+10x+3 = 8\left[x^2+\frac{10}{8}x+\frac{3}{8}\right] = 8\left[\left(x+\frac{5}{8}\right)^2-\frac{25}{64}+\frac{3}{8}\right] \qquad \frac{3}{8} = \frac{24}{64}$$

$$= 8\left[(x+5/8)^2 - 1/64\right] = 8\left(x+\frac{5}{8}-\frac{1}{8}\right)\left(x+\frac{5}{8}+\frac{1}{8}\right)$$

(d.) $8x^2+10x+3$,

$$8x^2+10x+3 = \boxed{(2x+1)(4x+3)}$$

Remark: we can derive this factorization via completing the square, That's what I'm showing

$$= 8\left(x+\frac{4}{8}\right)\left(x+\frac{6}{8}\right)$$
$$= 8(x+1/2)(x+3/4)$$
$$= 2(x+1/2)\,4(x+3/4)$$
$$= (2x+1)(4x+3)$$

(e.) $(x^2+10x+25)^2$,

$$(x^2+10x+25)^2 = \left((x+5)^2\right)^2 = \boxed{(x+5)^4}$$

(f.) $x^4-13x^2+36.$ $= (x^2-4)(x^2-9)$

$$= \boxed{(x-2)(x+2)(x-3)(x+3)}$$

Problem 7: (3pt) Solve the following polynomial equations. You can just write down the answers here since they should be immediately clear from your work on the previous problem.

(a.) $30x^3+15x^4=0$,

$$15x^3(x+2)=0 \Rightarrow \boxed{x=0} \text{ or } \boxed{x=-2}$$

(b.) $x^2-14x+48=0$,

$$(x-6)(x-8)=0 \Rightarrow \boxed{x=6} \text{ or } \boxed{x=8}$$

(c.) $2x^2+7x-4=0$,

$$(2x-1)(x+4)=0 \Rightarrow 2x=1 \text{ or } x=-4$$
$$\therefore \boxed{x=1/2} \text{ or } \boxed{x=-4}$$

(d.) $8x^2+10x+3=0$,

$$8\left(x+\frac{1}{2}\right)\left(x+\frac{3}{4}\right)=0 \Rightarrow \boxed{x=\frac{-1}{2}} \text{ or } \boxed{x=\frac{-3}{4}}$$

(e.) $(x^2+10x+25)^2=0$,

$$(x+5)^4=0 \Rightarrow \boxed{x=-5}$$

(f.) $x^4-13x^2+36=0.$

$$(x-2)(x+2)(x-3)(x+3)=0 \Rightarrow \boxed{x=2,-2,3,-3}$$

Problem 8: (5pt) For each quadratic polynomial $f(x)$ given below, complete the square and find all real or complex solutions of $f(x) = 0$:

(a.) $f(x) = x^2 + 6x + 13,$ $= (x+3)^2 - 9 + 13 = \underline{(x+3)^2 + 4}.$

$f(x) = 0 \longrightarrow (x+3)^2 = -4 \quad \therefore \quad x+3 = \pm\sqrt{-4} = \pm i\sqrt{4}$

$\boxed{x = -3 \pm 2i}$

Can also see sol$^{\underline{n}}$ by factoring below,

$\left((x+3)^2 + 4 = (x+3)^2 - (2i)^2 = (x+3-2i)(x+3+2i) = 0\right)$

(b.) $f(x) = x^2 - 8x + 16,$

$f(x) = (x-4)^2 - 16 + 16 \quad \therefore \quad \underline{f(x) = (x-4)^2}.$

$(x-4)^2 = 0$ has solution $\boxed{x = 4}$

(you could also say it has two sol$^{\underline{n}}$'s which are both $x = 4$, we do that sometimes)

(c.) $f(x) = x^2 + 3x - 3,$

$f(x) = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - 3 = \underline{\left(x + \frac{3}{2}\right)^2 - \frac{21}{4}}.$

$\left(x + \frac{3}{2}\right) - \frac{21}{4} = 0 \Rightarrow \left(x + \frac{3}{2}\right)^2 = \frac{21}{4}$

$\Rightarrow x + \frac{3}{2} = \pm\sqrt{\frac{21}{4}} \quad \therefore \quad \boxed{x = \frac{-3 \pm \sqrt{21}}{2}}$

(d.) $f(x) = 4x^2 - 16x + 15,$

$= 4\left(x^2 - 4x + \frac{15}{4}\right)$

$= 4\left((x-2)^2 - 4 + \frac{15}{4}\right)$

$= \underline{4\left[(x-2)^2 - \frac{1}{4}\right]}.$

Thus $f(x) = 0$ yields

$(x-2)^2 = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

$\Rightarrow x - 2 = \pm\sqrt{\frac{1}{4}} = \frac{\pm 1}{2}$

$\boxed{x = 2 \pm 1/2 = \frac{5}{2}, \frac{3}{2}}$

(e.) $f(x) = 2x^2 + 8x + 10.$

$= 2(x^2 + 4x + 5)$

$= 2\left((x+2)^2 - 4 + 5\right)$

$= \underline{2\left((x+2)^2 + 1\right)}.$

Thus $f(x) = 0$ yields,

$(x+2)^2 = -1$

$\Rightarrow x + 2 = \pm\sqrt{-1} = \pm i$

$\therefore \boxed{x = -2 \pm i}$

Problem 9: (4pt) Solve the following over $\mathbb{R}$,

(a.) $\dfrac{2x-1}{x+2} = \dfrac{4}{5}$,

$$(2x-1)(5) = 4(x+2)$$

$$10x - 5 = 4x + 8$$

$$6x = 13$$

$$\boxed{x = 13/6}$$

(b.) $\sqrt{5-x} + 1 = x - 2$,

$$\sqrt{5-x} = x - 3$$

$$\left(\sqrt{5-x}\right)^2 = (x-3)^2 = x^2 - 6x + 9$$

$$5 - x = x^2 - 6x + 9$$

$$x^2 - 5x + 4 = (x-1)(x-4) = 0$$

Thus $x = 1$ and $x = 4$ are potential sol^{n}'s.

<u>Check</u>: $\sqrt{5-1} + 1 = \sqrt{4} + 1 = 2 + 1 = \underbrace{3 \stackrel{?}{=} 1 - 2 = -1}_{\text{oops!}}$

$\sqrt{5-4} + 1 = 2 \stackrel{?}{=} 4 - 2 \stackrel{\checkmark}{=} 2$

We find $\boxed{x = 4}$ is the sol^{n} ($x = 1$ is extraneous)

(c.) $|3x + 5| = 11$.

$$3x + 5 = \pm 11$$

$$3x = -5 \pm 11$$

$$x = \frac{-5 \pm 11}{3} \longrightarrow x = \frac{-5+11}{3} \quad \text{or} \quad x = \frac{-5-11}{3}$$

aha, $\boxed{x = 2 \text{ or } x = \frac{-16}{3}}$

(d.) $\sqrt{1+x}+\sqrt{1-x}=2$,

$$\left(\sqrt{1+x}\right)^2=\left(2-\sqrt{1-x}\right)^2$$
$$=4-4\sqrt{1-x}+\left(\sqrt{1-x}\right)^2$$
$$=4-4\sqrt{1-x}+1-x$$
$$=5-x-4\sqrt{1-x}$$

Hence, $1+x=5-x-4\sqrt{1-x}$

which gives $4\sqrt{1-x}=4-2x$

$$\left(4\sqrt{1-x}\right)^2=(4-2x)^2$$
$$16(1-x)=16-16x+4x^2$$
$$4x^2-16x+16-16+16x=0$$
$$4x^2=0$$
$$\underline{x=0}. \quad \text{(neat)}$$

<u>Check</u>: $\sqrt{1+0}+\sqrt{1-0}=\sqrt{1}+\sqrt{1}=1+1\overset{\checkmark}{=}2$.

Problem 10: (1pt) Find real numbers a, b for which $a+ib=\dfrac{(7-i)(4+2i)}{(3-7i)^2}$.

$$\frac{(7-i)(4+2i)}{(3-7i)(3-7i)}=\frac{28+10i-2i^2}{9-42i+49i^2} \quad : \; i^2=-1$$
$$=\left[\frac{30+10i}{-40-42i}\right]\left[\frac{-40+42i}{-40+42i}\right]$$
$$=\frac{30(-40)+30(42)i+10(-40)i+420i^2}{(-40)^2+(42)^2}$$
$$=\frac{-1200-420+(1260-400)i}{1600+1764}$$
$$=\frac{-1620+860i}{3364}=\frac{-405}{841}+i\left(\frac{215}{841}\right)$$

$$\boxed{a=\frac{-405}{841}}$$
$$\boxed{b=\frac{215}{841}}$$