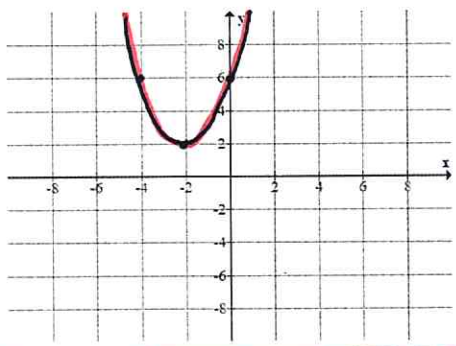
Please print this out and write your solutions on this document. I will only give half credit if the solutions are not written on this form. Please staple when finished. 60pts to earn here. Thanks!

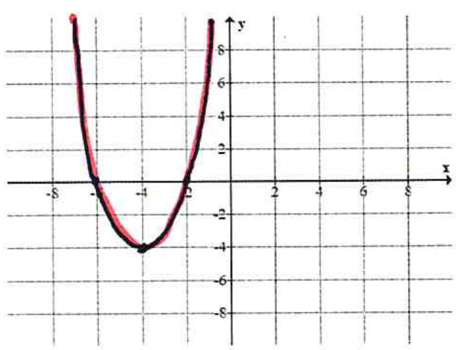
Problem 46: (6pts) *The discriminant for $f(x) = ax^2 + bx + c$ is $b^2 - 4ac$. Recall, non-negative discriminant implies the quadratic polynomial can be factored over $\mathbb{R}$ whereas $b^2 - 4ac < 0$ implies $ax^2 + bx + c$ cannot be factored over $\mathbb{R}$.*
Calculate the discriminant for each $f(x)$ given below and factor $f(x)$ over $\mathbb{R}$ if possible. In addition, graph $y = f(x)$ carefully in the plot provided:

(a.) $f(x) = x^2 + 4x + 6 = (x+2)^2 + 2$ (not possible to factor,
$b^2 - 4ac = 16 - 4(1)(6) = -8 <$



- From completing the square I can see $(-2, 2)$ is vertex
- Also, $f(0) = 6$ is y-intercept.

(b.) $f(x) = x^2 + 8x + 12 = (x+4)^2 - 4 = (x+4-2)(x+4+2) = (x+2)(x+6)$

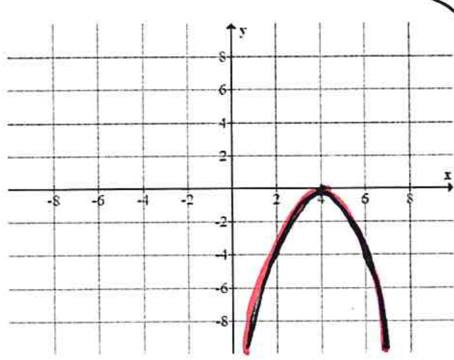


$b^2 - 4ac = 64 - 4(1)(12) = 16 > 0$
as you see above, we can factor $f(x)$ over $\mathbb{R}$.

- Also, completing square revealed $(-4, -4)$ is vertex.
- $f(0) = 12$ is y-intercept (out of range for graph)

(c.) $f(x) = -x^2 + 8x - 16$

$$\begin{aligned} f(x) &= -(x^2 - 8x + 16) \\ &= -((x-4)^2 - 16 + 16) \\ &= -(x-4)^2 \quad \text{yep.} \end{aligned}$$



Then vertex at $(4, 0)$ and this parabola opens down.

$b^2 - 4ac = 64 - 4(-1)(-16) = 64 - 64 = 0.$
discriminant zero for repeated root case.

Problem 47: (2pts) Find a polynomial of least degree whose graph crosses the x-axis at $x = -4$ and $x = 3$ and bounces off the x-axis at $x = 1$. In addition, assume the y-intercept is 20. Find the formula for $P(x)$.

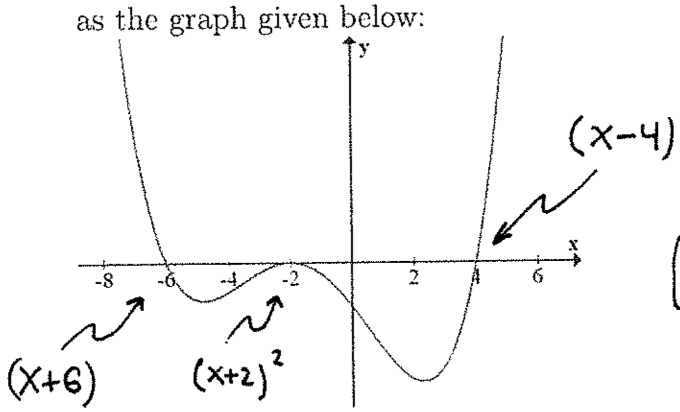
$$P(x) = \underset{\text{we can adjust.}}{\underset{\uparrow}{A}}\ \underbrace{(x+4)(x-3)}_{\text{odd power gives crossing}}\underbrace{(x-1)^2}_{\text{even gives bounce.}}$$

$$P(0) = 20 = A(4)(-3)(-1)^2 = -12A$$

$$A = \frac{20}{-12} = \frac{2\cdot 10}{-2\cdot 6} = \frac{2\cdot 2\cdot 5}{-2\cdot 2\cdot 3} = \frac{-5}{3}$$

$$\therefore \boxed{P(x) = \frac{-5}{3}(x+4)(x-3)(x-1)^2}$$

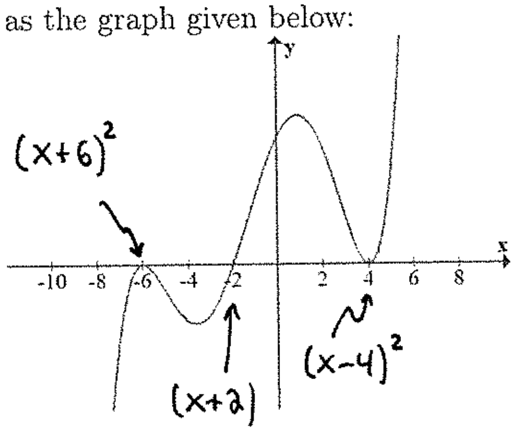
Problem 48: (2pts) Find $P(x)$ which could have a graph which shares the same shape and x-intercepts as the graph given below:



$$\boxed{P(x) = (x+6)(x+2)^2(x-4)}$$

($P(x) \approx x^4$ for $x \to \pm\infty$ and this matches given graph)

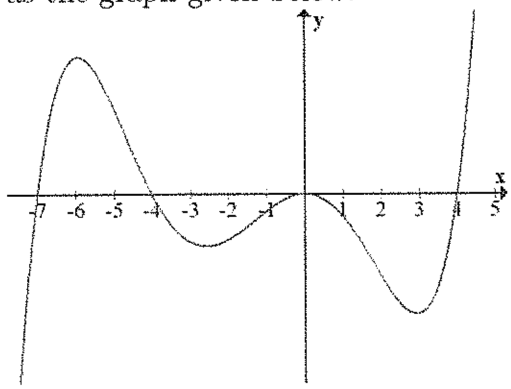
Problem 49: (1pts) Find $P(x)$ which could have a graph which shares the same shape and x-intercepts as the graph given below:



$$\boxed{P(x) = (x+6)^2(x+2)(x-4)^2}$$

($P(x) \approx x^5$ for $x \to \pm\infty$ and this matches given graph)

Problem 50: (1pts) Find $P(x)$ which could have a graph which shares the same shape and x-intercepts as the graph given below:



$$\boxed{P(x) = (x+7)(x+4)x^2(x-4)}$$

Problem 51: (2pts) Find a polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_o$ with zeros $-2, 0, 1, 3$ given that $a_3 = 4$.

$f(x) = A(x+2)x(x-1)(x-3)$ since $f(-2) = f(0) = f(1) = f(3) = 0$ and the factor theorem tells us $f(c) = 0 \iff (x-c)$ is factor.

Multiply out the above,

$$f(x) = Ax(x+2)[x^2 - 4x + 3]$$
$$= Ax[x^3 - 4x^2 + 3x + 2x^2 - 8x + 6]$$
$$= Ax^4 - 2Ax^3 - 5Ax^2 + 6Ax = a_4x^4 + 4x^3 + a_2x^2 + a_1x + a_o$$

Compare coefficients of x^3 to see $-2A = 4 \Rightarrow \underline{A = -2}$.

Thus, $\boxed{f(x) = -2x(x+2)(x-1)(x-3) = -2x^4 + 4x^3 + 10x^2 - 12x}$

Problem 52: (2pts) Let $P(x) = x^3 + 2x^2 - 9x - 18$. Show that -2 is a zero of $P(x)$ and find all the other zeros of $P(x)$. *Hint: factoring by grouping is a good idea here*

$$P(x) = x^2(x+2) - 9(x+2) = (x^2 - 9)(x+2)$$

$$\therefore \boxed{P(x) = (x-3)(x+3)(x+2)}$$

(Zeros of $P(x)$ are $3, -3$ and -2.)

Problem 53: (2pts) Let $f(x) = x^4 + 2x^2 - 3x + 10$. Use long division to calculate $\frac{f(x)}{x^2+3}$.

Is $(x^2 + 3)$ a factor of $f(x)$?

$$
\begin{array}{r}
x^2 - 1 \\
x^2+3 \;\overline{\big)\; x^4 + 2x^2 - 3x + 10} \\
-(x^4 + 3x^2) \\
\hline
-x^2 - 3x + 10 \\
-(-x^2 - 3) \\
\hline
\underbrace{-3x + 13}_{\text{remainder}}
\end{array}
$$

$$\Rightarrow \frac{x^4+2x^2-3x+10}{x^2+3} = x^2 - 1 + \frac{13-3x}{x^2+3}$$

(no, x^2+3 is not a factor of $f(x)$)

Another way to check,
if x^2+3 is factor of $f(x)$ then $f(i\sqrt{3}) = 0$. But,

$$f(i\sqrt{3}) = (i\sqrt{3})^4 + 2(i\sqrt{3})^2 - 3(i\sqrt{3}) + 10 \qquad i^4 = i^2 i^2 = (-1)(-1) = 1.$$

$$= 9 - 2(3) - 3i\sqrt{3} + 10 \neq 0 \quad \therefore \; x^2+3 \text{ not a factor.}$$

Problem 54: (2pts) Let $f(x) = x^5 + 12x^2 - 3x + 2$. Calculate $\frac{f(x)}{x-1}$.

Is $(x - 1)$ a factor of $f(x)$?

$$
\begin{array}{r}
x^4 + x^3 + x^2 + 13x + 10 \\
x - 1 \;\overline{\big)\; x^5 + 12x^2 - 3x + 2} \\
-(x^5 - x^4) \\
\hline
x^4 + 12x^2 - 3x + 2 \\
-(x^4 - x^3) \\
\hline
x^3 + 12x^2 - 3x + 2 \\
-(x^3 - x^2) \\
\hline
13x^2 - 3x + 2 \\
-(13x^2 - 13x) \\
\hline
10x + 2 \\
-(10x - 10) \\
\hline
\boxed{12}
\end{array}
$$

remainder, nonzero $\therefore$ $\boxed{(x-1) \text{ not factor of } f(x)}$

Notice, $f(1) = 1 + 12 - 3 + 2 = 12 \neq 0$
thus by factor Th$^{\underline{m}}$ $(x-1)$ not factor. (also, notice this illustrates the remainder theorem!)

Problem 55: (2pts) Factor $f(x) = x^5 - 3x^4 - 2x^3 + 6x^2 - 3x + 9$ completely over $\mathbb{R}$. Hint: $f(3) = 0$.

$$\left.\begin{aligned} f(x) &= x^4(x-3) - 2x^2(x-3) - 3(x-3) \\ &= (x^4 - 2x^2 - 3)(x-3) \end{aligned}\right\}$$ factored by grouping, could use long division alternatively.

$$= (x^2 - 3)(x^2+1)(x-3)$$

$$= \boxed{(x - \sqrt{3})(x + \sqrt{3})(x^2+1)(x-3)}$$

Problem 56: (2pts) Find the standard form of a polynomial with real coefficients of degree 4 which has complex zeros $3 + 2i$ and $7 - 3i$ with a y-intercept of 10.

$f(3+2i) = 0 \implies (x-3)^2 + 4$ factors the polynomial

$f(7-3i) = 0 \implies (x-7)^2 + 9$ factors the polynomial

Since the polynomial is degree 4 we find,

$$P(x) = A(x^2 - 6x + 13)(x^2 - 14x + 58)$$

Then $P(0) = A(13)(58) = 10 \implies A = \dfrac{10}{13(58)} = \dfrac{5}{377}$

Hence $P(x) = \dfrac{5}{377}(x^4 - 20x^3 + 155x^2 - 530x + 754)$

$$\boxed{P(x) = \frac{5}{377}x^4 - \frac{100}{377}x^3 + \frac{775}{337}x^2 - \frac{2650}{377}x + 10}$$

Problem 57: (2pts) Factor $f(x) = x^4 + 7x^3 + 19x^2 + 23x + 10$ completely over $\mathbb{R}$. Hint: $f(-2+i) = 0$.

$f(-2+i) = 0 \Rightarrow (x+2)^2 + 1 = x^2 + 4x + 5$ factors $f(x)$.

$$\begin{array}{r}
x^2 + 3x + 2 \\
x^2+4x+5 \,\big)\, \overline{x^4 + 7x^3 + 19x^2 + 23x + 10} \\
-(x^4 + 4x^3 + 5x^2) \\
\hline
3x^3 + 14x^2 + 23x + 10 \\
-(3x^3 + 12x^2 + 15x) \\
\hline
2x^2 + 8x + 10 \\
-(2x^2 + 8x + 10) \\
\hline
\boxed{0}
\end{array}$$

$f(x) = (x^2 + 4x + 5)(x^2 + 3x + 2)$

$\Rightarrow \boxed{f(x) = (x+1)(x+2)(x^2 + 4x + 5)}$

Problem 58: (2pts) State the rational roots theorem in your own words.

Possible rational zeros for a polynomial are found from ratios of the factors of the constant coeff. and the leading coefficient.

Problem 59: (2pts) If $R(x) = 2x^5 + 3x^3 + 4x^2 - 8$ then use the Rational Roots Theorem (aka the Rational Zeros Theorem) to list all possible rational zeros for $R(x)$.

-8 has factors $\pm 1, \pm 2, \pm 4, \pm 8$

2 has factors $\pm 1, \pm 2$

$\Rightarrow \frac{\pm 1}{1}, \frac{\pm 1}{2}, \frac{\pm 2}{1}, \frac{\pm 2}{2}, \frac{\pm 4}{1}, \frac{\pm 4}{2}, \frac{\pm 8}{1}, \frac{\pm 8}{2}$

That is, $\boxed{\pm 1, \pm 1/2, \pm 2, \pm 4, \pm 8}$

Problem 60: (2pts) It is known that $P(x) = x^3 + 2x^2 - 13x + 10$ has real zeros which are integers. Factor $P(x)$ completely. *Hint: use the Rational Roots Theorem*

10 has factors $\pm 1, \pm 2, \pm 5, \pm 10$ these are the possible roots.

$P(1) = 1 + 2 - 13 + 10 = 0$, $P(2) = 8 + 8 - 26 + 10 = 0$, $P(-5) = 0$

$\therefore \boxed{P(x) = (x-1)(x-2)(x+5)}$

Remark: sorry about typo here!

Problem 61: (2pts) Use Descartes' rule of signs to determine how many positive and how many negative real zeros there are for the polynomial $P(x) = 2x^6 + 5x^4 - x^3 - 5x - 1$.

$+ \quad - \quad - \quad -$ (one variation)

$P(-x) = 2x^6 + 5x^4 + x^3 + 5x - 1$

$+ \quad + \quad + \quad -$ (one variation)

- $P(x)$ has <u>one</u> positive real zero.
- $P(x)$ has <u>one</u> negative real zero.

Problem 62: (3pts) Factor the following polynomials completely over the complex numbers.

(a.) $x^2 - 4x + 5 = (x-2)^2 + 1 = (x-2+i)(x-2-i)$

(b.) $x^4 + 4x^2 - 36 = (x^2+2)^2 - 40$

$= (x^2 + 2 - \sqrt{40})(x^2 + 2 + \sqrt{40})$

$= (x - \sqrt{\sqrt{40}-2})(x + \sqrt{\sqrt{40}-2})(x + i\sqrt{\sqrt{40}+2})(x - i\sqrt{\sqrt{40}+2})$

(c.) $x^4 + x^2 = x^2(x^2+1)$

$= \boxed{x^2(x-i)(x+i)}$

Sorry ☺ I meant to put $x^4 + 5x^2 - 36$ which is much nicer,

$x^4 + 5x^2 - 36 = (x^2+9)(x^2-4) = (x+3i)(x-3i)(x-2)(x+2)$

Problem 63: (2pts) Let $f(x) = (x^2 - 4)(x^2 - x - 2)^2$. Find all zeros of $f(x)$ and determine the multiplicity of each zero.

$f(x) = (x-2)(x+2)[(x-2)(x+1)]^2$

$= (x-2)(x+2)(x-2)^2(x+1)^2$

$= (x+2)(x-2)^3(x+1)^2$

$f(-2) = 0$ and $(x+2)$ has multiplicity 1.

$f(2) = 0$ has multiplicity 3.

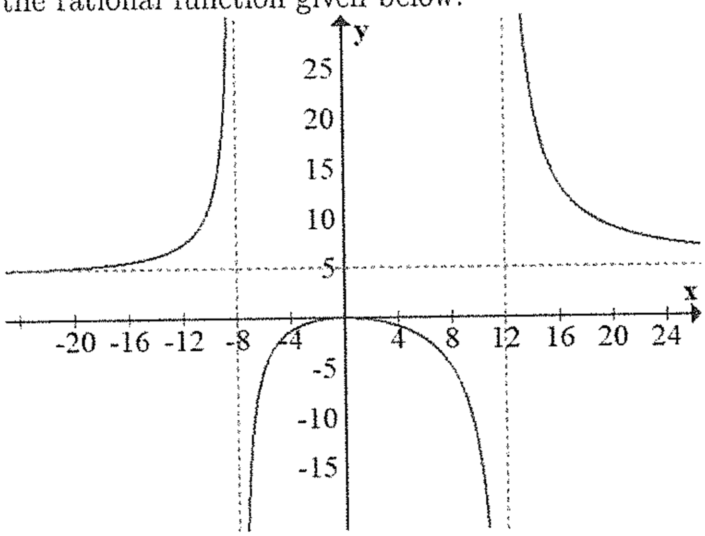
$f(-1) = 0$ has multiplicity 2.

Problem 64: (2pts) Find the x and y-intercepts of $f(x) = \dfrac{x^2 - x - 2}{x - 6} = \dfrac{(x-2)(x+1)}{x-6}$

x-intercepts: $(x-2)(x+1) = 0 \Rightarrow \underline{x = 2}$ & $\underline{x = -1}$

y-intercept: $f(0) = \dfrac{-2}{-6} = \dfrac{1}{3}$.

Problem 65: (2pts) Write the equations for each horizontal and vertical asymptote for the graph of the rational function given below:



V.A. at $x = -8$ and $x = 12$

H.A. is $y = 5$.

Problem 66: (4pts) Consider the rational function $f(x) = \dfrac{x^2+4x-5}{x^2+x-2}$. Find all vertical or horizontal asymptotes, as well as any holes in the graph. Graph the function carefully with each feature clearly labeled.

$$f(x) = \frac{x^2+4x-5}{x^2+x-2} = \frac{(x+5)(x-1)}{(x+2)(x-1)}$$

x-intercept at $x=-5$
VA at $x=-2$
Hole at $x=1$

$$f_{reduced}(x) = \frac{x+5}{x+2} \quad \text{thus} \quad f_{reduced}(1) = \frac{1+5}{1+2} = \frac{6}{3} = 2$$

the hole in graph is at $(1, 2)$.

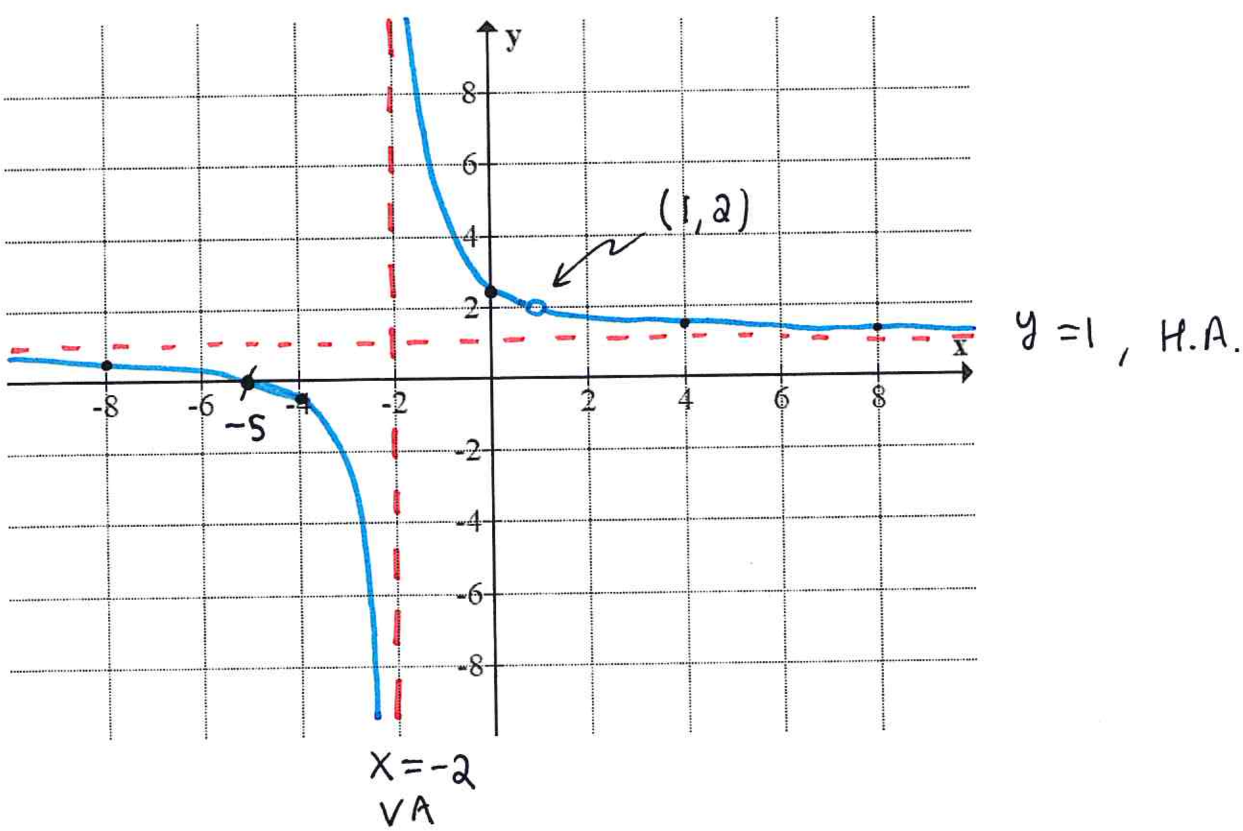
Also, $f(0) = \frac{-5}{-2} = 2.5$ gives y-intercept.

HA is $y=1$ as $x \to \pm\infty$.

A few additional data points help,

$$f(-8) = \frac{-8+5}{-8+2} = \frac{-3}{-6} = \frac{1}{2} \quad \& \quad f(-4) = \frac{-4+5}{-4+2} = \frac{1}{-2}$$

$$f(4) = \frac{4+5}{4+2} = \frac{9}{6} = 1.5 \quad \& \quad f(8) = \frac{8+5}{8+2} = \frac{13}{10} = 1.3$$



Problem 67: (4pts) Consider the rational function

$$f(x) = 2 + \frac{(x-1)(x^2-6x+9)}{(x^2-2x+1)(x-3)(x^2-16)}.$$

Find all vertical or horizontal asymptotes, as well as any holes in the graph. Graph the function carefully with each feature clearly labeled.

$$f(x) = 2 + \frac{(x-1)(x-3)^2}{(x-1)^2(x-3)(x-4)(x+4)} = 2 + \frac{x-3}{(x-1)(x-4)(x+4)}$$

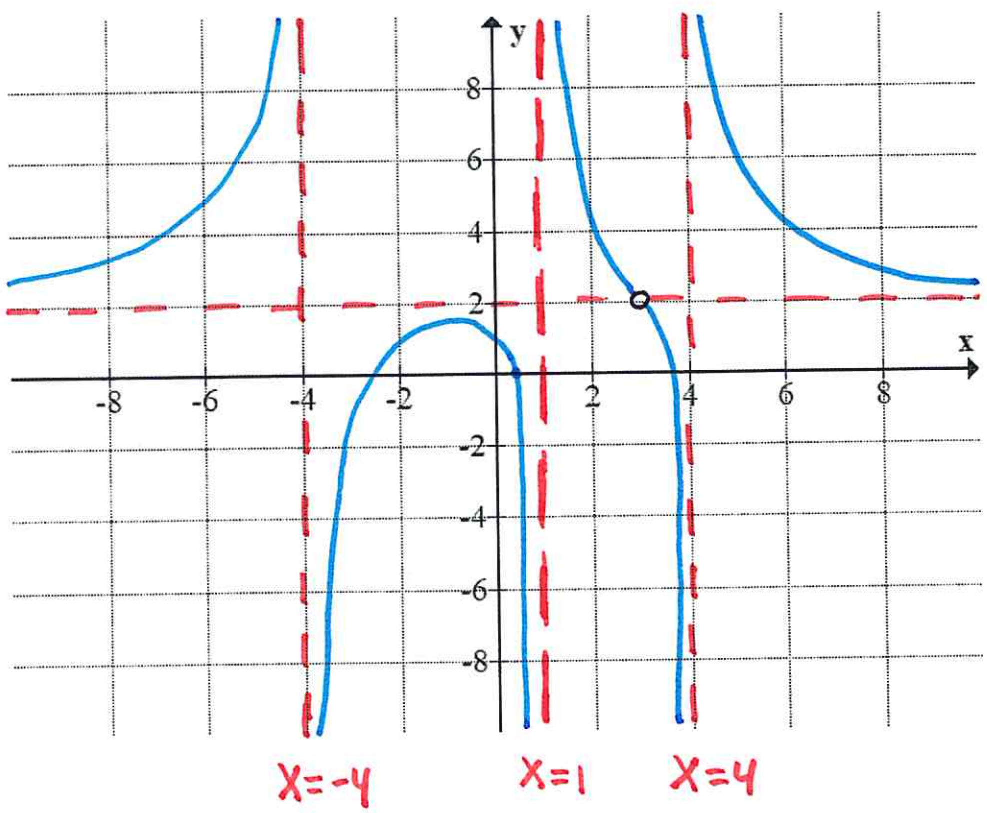
We find H.A. of $y=2$, VA at $x = 1, -4, 4$ and hole in graph at $x=3$ where $f_{red}(3) = 2 \Rightarrow (3,2)$ is hole.

To find x-intercepts we need to do more algebra,

$$f(x) = \frac{2(x-1)(x-4)(x+4) + x-3}{(x-1)(x-4)(x+4)}$$

$$= \frac{2(x-1)(x^2-16) + x - 3}{(x-1)(x^2-16)} = \frac{2x^3-2x^2-31x+29}{(x-1)(x^2-16)}$$

Observe $g(x) = 2x^3-2x^2-31x+29$ has $g(0) = 29$ & $g(1) = -2$ thus there exists an x-intercept for $f(x)$ between $x=0$ and $x=1$.



(I knew to look for this from classwork on 10/22/2020)

$f(-8) > 2$

$f(-2) < 2$

$f(6) > 2$

Problem 68: (2pts) Solve $x^3 + 4x^2 \geq 4x + 16$. Write the answer in interval notation.

$$\underbrace{x^3 + 4x^2 - 4x - 16}_{f(x) \,=\, x^2(x+4) - 4(x+4) \,=\, (x^2-4)(x+4) \,=\, (x-2)(x+2)(x+4)} \geq 0$$

$$f(x) = x^2(x+4) - 4(x+4) = (x^2-4)(x+4) = \underbrace{(x-2)(x+2)(x+4)}$$

algebraic critical # of $x = 2, -2, -4$

$$- - - \;|\; + + + + \;|\; - - - \;|\; + + + \rightarrow$$
$$-4 \qquad -2 \qquad 2$$

$$\Rightarrow \boxed{[-4, -2] \cup [2, \infty)}$$

Problem 69: (3pts) Solve $\dfrac{x-3}{2x+5} \geq 1$. Write the answer in interval notation.

$$\frac{x-3}{2x+5} - 1 \geq 0$$

$$\frac{x-3-(2x+5)}{2x+5} \geq 0$$

$$f(x) = \frac{-x-8}{2x+5} \geq 0 \Rightarrow$$

$x = -8$ & $x = -5/2$ are places where the expression may change sign.

$$- - - \;|\; + + + + \;|\; - - - \rightarrow$$
$$-8 \qquad -5/2$$

$f(0) = \frac{-8}{5} < 0$

$f(-3) = \frac{-5}{-1} = 5 > 0$

thus $\boxed{[-8, -5/2)}$

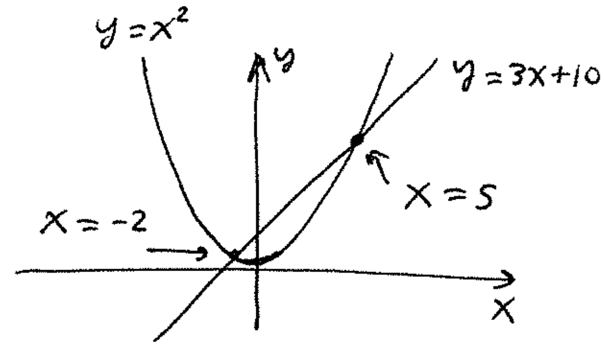
Problem 70: (2pts) Find all x for which the graph $f(x) = x^2$ lies above the graph of $g(x) = 3x + 10$.

We want $f(x) \geq g(x)$

$$x^2 \geq 3x + 10$$

$$x^2 - 3x - 10 \geq 0$$

$$f(x) = (x-5)(x+2) \geq 0$$



$+++ \mid ---- \mid +++ \rightarrow f(x)$

$-2 \qquad 5$

$\therefore \boxed{(-\infty, -2] \cup [5, \infty)}$

Problem 71: (2pts) Find the domain of $h(x) = \sqrt[4]{x^4 - 1}$.

We desire $x^4 - 1 \geq 0$

$$(x^2+1)(x^2-1) \geq 0$$

$$f(x) = (x^2+1)(x-1)(x+1) \geq 0$$

$$f(0) = -1 < 0$$

$+++ \mid --- \mid +++ \rightarrow f(x)$

$-1 \qquad 1$

$\Rightarrow \boxed{\text{dom}(h(x)) = (-\infty, -1] \cup [1, \infty)}$