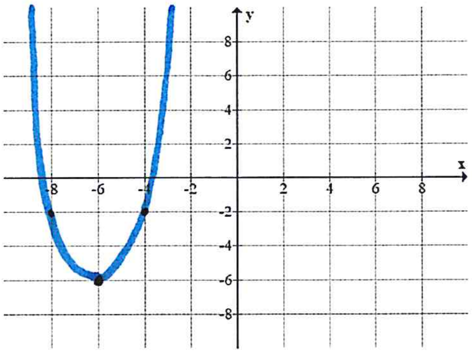
NAME: ______________________________.

MATH 113: **TEST 3**

No phones. You are allowed a calculator and a sheet of notes front and back. At least 150pts to earn here. Thanks!

Problem 1: (10pts) Factor $f(x) = x^2 + 12x + 30$ over $\mathbb{R}$ if possible, find the vertex of the parabola $y = f(x)$, and graph $y = f(x)$ carefully in the plot provided:



$$f(x) = x^2 + 12x + 30$$
$$= (x+6)^2 - 36 + 30$$
$$= (x+6)^2 - 6$$
$$= \underline{(x+6-\sqrt{6})(x+6+\sqrt{6})} \text{ factored.}$$

vertex at (-6, -6).

$$f(-4) = 2^2 - 6 = -2 = f(-8)$$

Problem 2: (15pts) Solve $\dfrac{1}{x^2-9} > 0$ and write your answer using interval notation.

$x^2 - 9 = (x-3)(x+3)$ ⇒ ±3 are possible places for sign change (algebraic critical #)

$$\underset{}{+++} \;\big|_{-3}\; - - - \;\big|_{3}\; +++ \rightarrow$$

$$\boxed{(-\infty, -3) \cup (3, \infty)}$$

Problem 3: (15pts) Solve $\dfrac{x}{x+9} \leq 1$. Write the answer in interval notation.

$$\frac{x}{x+9} - 1 \leq 0 \Rightarrow \frac{x-(x+9)}{x+9} = \frac{-9}{x+9} \leq 0$$

$$+ + + + \;\big\|_{-9}\; - - - - \rightarrow$$

$$\boxed{(-9, \infty)}$$

Problem 4: (15pts) Suppose a polynomial $P(x)$ has a graph which crosses the x-axis at $x = -7$ and bounces off the x-axis at $x = 3$. Find formula of $P(x)$ given that the y-intercept is 42.

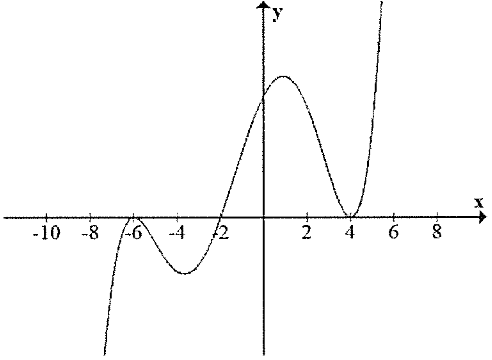
$$P(x) = A(x+7)(x-3)^2$$

$$P(0) = 42 = A(0+7)(0-3)^2 = 63A$$

$$A = \frac{42}{63} = \frac{3(14)}{3(21)} = \frac{2\cdot 7}{3\cdot 7} = \frac{2}{3}$$

$$\boxed{P(x) = \frac{2}{3}(x+7)(x-3)^2}$$

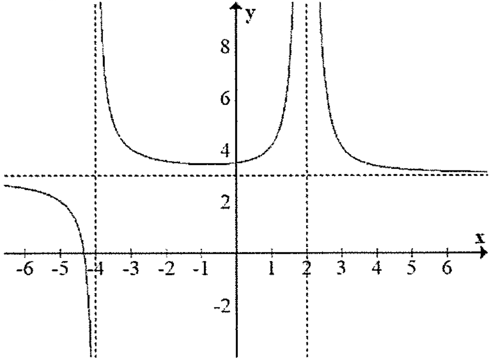
Problem 5: (15pts) Find $P(x)$ which could have a graph which shares the same shape and x-intercepts as the graph given below:



$$\boxed{P(x) = (x+6)^2(x+2)(x-4)^2}$$

$\left(P(x) \approx x^5 \text{ for } |x| >> 0 \right.$
matches given graph)

Problem 6: (10pts) Find a rational function $f(x)$ which could have a graph which shares the same shape as well as matching horizontal and vertical asymptotes of the graph given below:



$$\boxed{f(x) = \frac{1}{x+4} + \frac{1}{(x-2)^2} + 3}$$

$$f(x) = \frac{(x-2)^2 + x+4 + 3(x+4)(x-2)^2}{(x+4)(x-2)^2}$$

$$f(x) = \frac{x^2 - 4x+4 + x+4 + 3(x+4)(x^2-4x+4)}{(x+4)(x-2)^2}$$

$$f(x) = \frac{3x^3 + x^2 - 39x + 56}{(x-2)^2(x+4)}$$

Problem 7: (15pts) Let $P(x) = x^5 + 2x^4 - 81x - 162$. Show that -2 is a zero of $P(x)$ and factor $P(x)$ completely over $\mathbb{R}$.

$$\begin{aligned} P(-2) &= (-2)^5 + 2(-2)^4 - 81(-2) - 162 \\ &= -32 + 32 + 162 - 162 \\ &= 0. \end{aligned}$$

$$\begin{aligned} P(x) &= x^4(x+2) - 81(x+2) \quad : \text{factored by grouping.} \\ &= (x^4 - 81)(x+2) \\ &= (x^2-9)(x^2+9)(x+2) \\ &= \boxed{(x-3)(x+3)(x+2)(x^2+9)} \end{aligned}$$

Problem 8: (15pts) Factor $f(x) = x^5 - 9x^4 + 37x^3 - 67x^2 + 54x - 16$ completely over $\mathbb{R}$. Hint: $f(\underbrace{3 + i\sqrt{7}}) = 0$.

$$(x-3)^2 + (\sqrt{7})^2 = x^2 - 6x + 9 + 7 = x^2 - 6x + 16 \text{ is factor.}$$

$$\begin{array}{r} x^3 - 3x^2 + 3x - 1 \\ x^2 - 6x + 16 \overline{)\, x^5 - 9x^4 + 37x^3 - 67x^2 + 54x - 16} \\ -(x^5 - 6x^4 + 16x^3) \\ \hline -3x^4 + 21x^3 - 67x^2 + 54x - 16 \\ -(-3x^4 + 18x^3 - 48x^2) \\ \hline 3x^3 - 19x^2 + 54x - 16 \\ (3x^3 - 18x^2 + 48x) \\ \hline -x^2 + 6x + 16 \\ -x^2 + 6x + 16 \\ \hline 0 \end{array}$$

$$\begin{aligned} f(x) &= (x^2 - 6x + 16)(x^3 - 3x^2 + 3x - 1) \\ &= \boxed{(x^2 - 6x + 16)(x-1)^3} \end{aligned}$$

Problem 9: (15pts) It is known that $P(x) = x^4 - 16x^3 + 86x^2 - 176x + 105$ has real zeros which are integers. Factor $P(x)$ completely. *Hint: use the Rational Roots Theorem;* $105 = 3 \cdot 5 \cdot 7$

$$P(3) = 81 - 16(27) + 86(9) - 176(3) + 105 = 0$$

$$P(5) = 625 - 16(125) + 86(25) - 176(5) + 105 = 0$$

$$P(7) = 7^4 - 16 \cdot 7^3 + 86(49) - 176(3) + 105 = 0$$

$$P(1) = 1 - 16 + 86 - 176 + 105 = 0$$

$$\boxed{P(x) = (x-1)(x-3)(x-5)(x-7)}$$

Problem 10: (20pts) Factor the following polynomials completely over the complex numbers.

(a.) $x^4 - 7x^3 + 9x^2$

$$\begin{aligned} &= x^2(x^2 - 7x + 9) \\ &= x^2\left(\left(x - \tfrac{7}{2}\right)^2 - \tfrac{49}{4} + \tfrac{36}{4}\right) \\ &= x^2\left(\left(x - \tfrac{7}{2}\right)^2 - \tfrac{13}{4}\right) \\ &= \boxed{x^2\left(x - 7/2 - \sqrt{13}/2\right)\left(x - \tfrac{7}{2} + \sqrt{13}/2\right)} \end{aligned}$$

(b.) $x^4 - 7x^2 - 8$

$$\begin{aligned} &= (x^2 - 8)(x^2 + 1) \\ &= \boxed{(x - \sqrt{8})(x + \sqrt{8})(x - i)(x + i)} \end{aligned}$$

Problem 11: (10pts) Consider the rational function $f(x) = \dfrac{2x^3}{16x - x^3}$. Find all vertical or horizontal asymptotes, as well as any holes in the graph. Graph the function carefully with each feature clearly labeled.

$$\frac{2x^3}{x(16-x^2)} = \frac{2x^3}{x(4-x)(4+x)} = \frac{-2x^2}{(x-4)(x+4)} \quad \text{for } x \neq 0, 4, -4$$

Hole at (0,0) where we bounce.

V.A. at $x = 4, -4$, cross-axis past V.A. since odd,

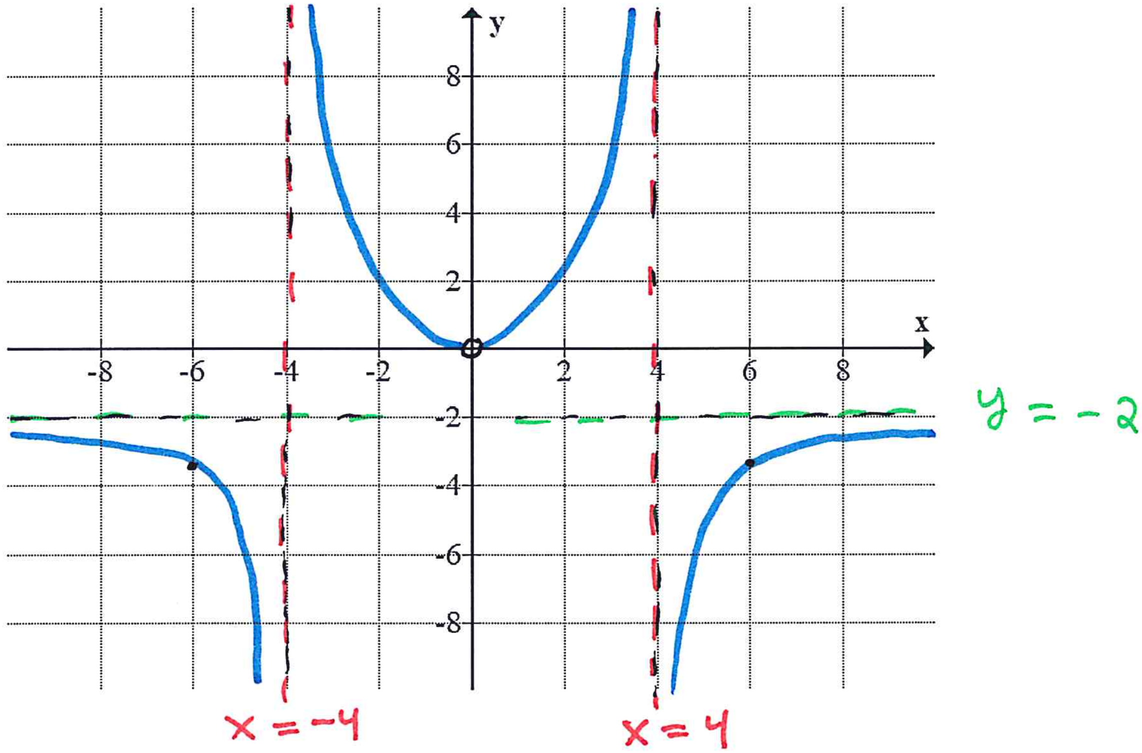
$$f(2) = \frac{-2(4)}{-2(6)} = \frac{-8}{-12} = \frac{2}{3}$$

$$f(-2) = \frac{-2(4)}{-6(2)} = \frac{2}{3}$$

$$f(6) = \frac{-2(36)}{2(10)} = -3.6$$

$$f(-6) = \frac{-2(36)}{-10(-2)} = -3.6$$

H.A. of $y = -2$



Problem 12: (5pts) Write the range of function in the previous problem in interval notation.

$$\boxed{\text{range}(f(x)) = (-\infty, -2) \cup (0, \infty)}$$