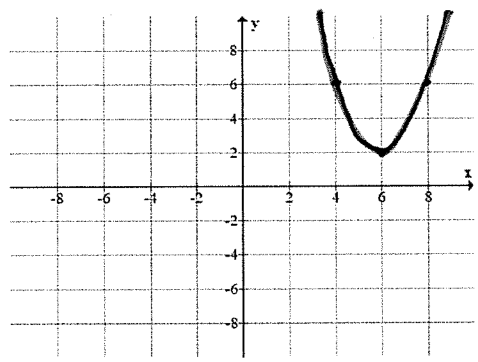
NAME: ________________________________.

MATH 113:	TEST 3

No phones. You are allowed a calculator and a sheet of notes front and back. At least 150pts to earn here. Thanks!

Problem 1: (10pts) Factor $f(x) = x^2 - 12x + 38$ over $\mathbb{R}$ if possible, find the vertex of the parabola $y = f(x)$, and graph $y = f(x)$ carefully in the plot provided:



$f(x) = (x-6)^2 + 2$

thus $\boxed{(6, 2) \text{ is vertex}}$

$f(8) = (8-6)^2 + 2 = 4 + 2 = 6$

$f(4) = f(8) = 6$

$f(2) = (2-6)^2 + 2 = 18 = f(10)$

$f(x) = (x-6)^2 + 2 = \underbrace{(x - 6 + i\sqrt{2})(x - 6 - i\sqrt{2})}_{\text{not possible using only real \#'s.}}$ (cannot factor over $\mathbb{R}$)

Problem 2: (15pts) Solve $\dfrac{1}{x^2 + 2x - 3} > 0$ and write your answer using interval notation.

$x^2 + 2x - 3 = (x+3)(x-1) \Rightarrow \dfrac{1}{x^2+2x-3}$ has $x = -3, 1$ as possible places where sign may flip.

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-3   1

$\Rightarrow \dfrac{1}{x^2+2x-3} > 0$ for $x \in \boxed{(-\infty, -3) \cup (1, \infty)}$

Problem 3: (15pts) Solve $\dfrac{x+4}{x-3} \leq 1$. Write the answer in interval notation.

$\dfrac{x+4}{x-3} - 1 \leq 0 \Rightarrow \dfrac{x+4-(x-3)}{x-3} = \dfrac{7}{x-3} \leq 0$

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3

cannot allow $x = 3$ since $\Rightarrow$ division by zero for $\dfrac{7}{x-3}$.

Thus $\boxed{(-\infty, 3)}$ solves $\dfrac{x+4}{x-3} \leq 1$.

Problem 4: (15pts) Suppose a polynomial $P(x)$ has a graph which crosses the x-axis at $x = -4$ and bounces off the x-axis at $x = 5$. Find formula of $P(x)$ given that the y-intercept is -200.

$P(x) = A(x+4)(x-5)^2$ and $P(0) = -200$

$P(0) = A \cdot 4 \cdot (-5)^2 = 100A = -200 \quad \therefore \quad A = \frac{-200}{100} = -2.$

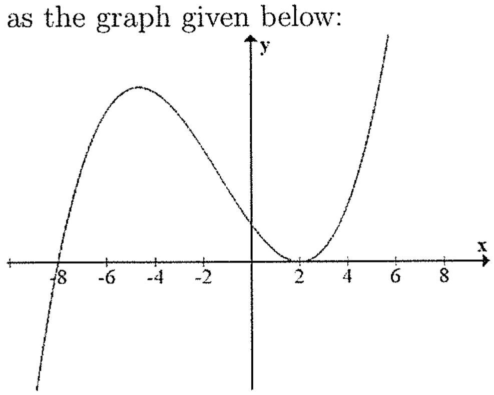
$$\boxed{P(x) = -2(x+4)(x-5)^2}$$

In standard form, $(x+4)(x^2-10x+25) = x^3 - 10x^2 + 25x + 4x^2 - 40x + 100$, we have,

$$\underline{P(x) = -2x^3 - 12x^2 + 30x - 200}.$$

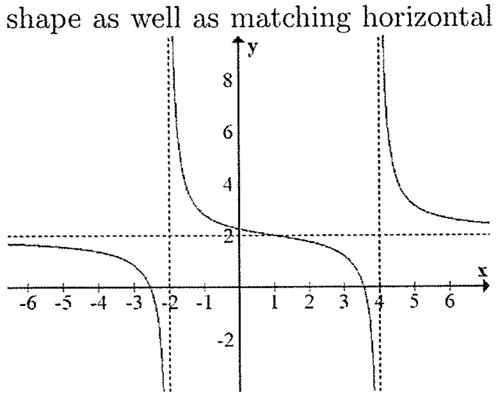
Remark: if I ask for standard form of $P(x)$ then more work is needed.

Problem 5: (15pts) Find $P(x)$ which could have a graph which shares the same shape and x-intercepts as the graph given below:



$P(x) = (x+8)(x-2)^2$ will do nicely, notice $P(x) \approx x^3$ for $|x| \gg 0$ as matches given graph, and $P(x)$ has $\underset{\text{crosses}}{\underline{P(-8)=0}}$ and $\underset{\text{with bounce.}}{\underline{P(2)=0}}$

Problem 6: (10pts) Find a rational function $f(x)$ which could have a graph which shares the same shape as well as matching horizontal and vertical asymptotes of the graph given below:



VA: $x = -2$ & $x = 4$

H.A.: $y = 2$

$$\boxed{f(x) = \frac{2(x-3.5)(x+2.5)}{(x+2)(x-4)}}$$

(I'll award partial credit even if x-intercepts are way off.)

Problem 7: (15pts) Let $P(x) = x^4 + 2x^3 - 5x^2 - 6x$. Show that 2 is a zero of $P(x)$ and factor $P(x)$ completely over $\mathbb{R}$.

$$P(2) = 2^4 + 2(2)^3 - 5(2)^2 - 6(2) = 16 + 16 - 20 - 12 = 0.$$

$$\begin{array}{r|l} & x^3 + 4x^2 + 3x \\ \hline x-2 & x^4 + 2x^3 - 5x^2 - 6x \\ & -(x^4 - 2x^3) \\ & \quad 4x^3 - 5x^2 - 6x \\ & \quad -(4x^3 - 8x^2) \\ & \qquad 3x^2 - 6x \\ & \qquad -(3x^2 - 6x) \\ & \qquad\qquad 0. \end{array}$$

$$\begin{aligned} P(x) &= (x-2)(x^3 + 4x^2 + 3x) \\ &= x(x-2)(x^2+4x+3) \\ &= \boxed{x(x-2)(x+1)(x+3)} \end{aligned}$$

Problem 8: (15pts) Factor $f(x) = x^4 - 2x^3 + x^2 + 18x - 90$ completely over $\mathbb{R}$. Hint: $f(1+3i) = 0$.

$f(1+3i) = 0 \Rightarrow (x-1)^2 + 9$ factors $f(x)$

$\Rightarrow x^2 - 2x + 10$ factors $f(x)$

$$\begin{array}{r|l} & x^2 - 9 \\ \hline x^2 - 2x + 10 & x^4 - 2x^3 + x^2 + 18x - 90 \\ & -(x^4 - 2x^3 + 10x^2) \\ & \quad -9x^2 + 18x - 90 \\ & \quad -(-9x^2 + 18x - 90) \\ & \qquad\qquad 0. \end{array}$$

$$\begin{aligned} f(x) &= (x^2 - 2x + 10)(x^2 - 9) \\ &= \boxed{(x-3)(x+3)(x^2-2x+10)} \end{aligned}$$

Problem 9: (15pts) It is known that $P(x) = x^3 - 7x^2 + 7x + 15$ has real zeros which are integers. Factor $P(x)$ completely. *Hint: use the Rational Roots Theorem*

$15 = 1 \cdot 3 \cdot 5 \Rightarrow \pm 1, \pm 3, \pm 5, \pm 15$ possible roots.

$P(1) = 1 - 7 + 7 + 15 = 16 \neq 0$

$P(-1) = -1 - 7 - 7 + 15 = -15 + 15 = 0 \Rightarrow (x+1)$ is factor of $P(x)$.

$P(3) = 27 - 7(9) + 7(3) + 15 = 15 - 63 + 48 = 63 - 63 = 0$

$\Rightarrow (x-3)$ factors $P(x)$.

$P(5) = 125 - 7(25) + 7(5) + 15 = 0 \Rightarrow (x-5)$ is factor of $P(x)$

$\therefore \boxed{P(x) = (x+1)(x-3)(x-5)}$

Check on work: $1(-3)(-5) = 15$.

Problem 10: (20pts) Factor the following polynomials completely over the complex numbers.

(a.) $x^4 - 6x^3 + 11x^2 = x^2(x^2 - 6x + 11)$ completed square.

$= x^2((x-3)^2 + 2)$

$= \boxed{x^2(x - 3 - i\sqrt{2})(x - 3 + i\sqrt{2})}$

(b.) $x^4 - 8x^2 - 7 = (x^2 - 7)(x^2 + 1)$ my intended problem. Oops!... Sorry! Should have given $x^4 - 6x^2 - 7$.

$(x^2 - 4)^2 - 16 - 7 = (x^2 - 4)^2 - 23$

$= (x^2 - 4 + \sqrt{23})(x^2 - 4 - \sqrt{23})$

$= \underbrace{(x - i\sqrt{\sqrt{23} - 4})(x + i\sqrt{\sqrt{23} - 4})(x - \sqrt{4 + \sqrt{23}})(x + \sqrt{4 + \sqrt{23}})}_{\text{worth bonus if you got it.}}$

Problem 11: (10pts) Consider the rational function $f(x) = \frac{2x^2}{16x - x^3}$. Find all vertical or horizontal asymptotes, as well as any holes in the graph. Graph the function carefully with each feature clearly labeled.

$$f(x) = \frac{2x^2}{x(16-x^2)} = \frac{2x^2}{x(4-x)(4+x)} = \underbrace{\frac{2x}{(4-x)(4+x)}}_{\text{freduced}}$$

- hole at $(0,0)$
- V.A. $x = \pm 4$
- H.A. $y = -2$
- H.A. $y = 0$

$\underbrace{\hphantom{xxxxxxxx}}$

degree numerator $= 2.$

degree denominator $= 3.$

$$f(2) = \frac{2(2)}{16-4} = \frac{4}{12} = \frac{1}{3}$$

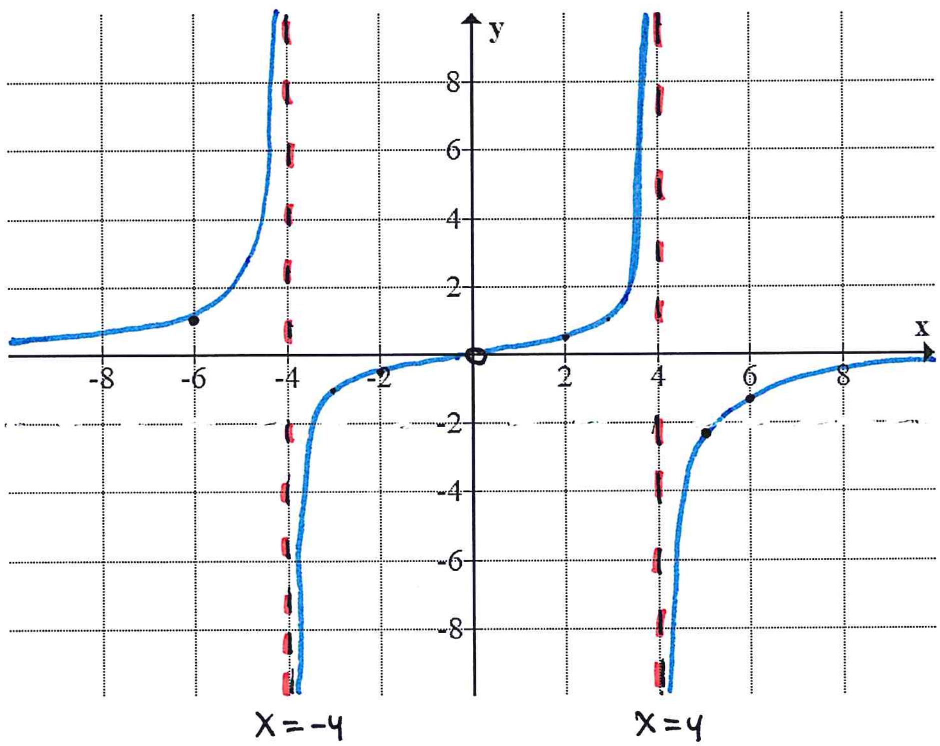
$$f(-2) = \frac{-4}{12} = \frac{-1}{3}$$

$$f(3) = \frac{2(3)}{16-9} = \frac{6}{7}$$

$$f(-6) = \frac{-12}{16-36} = \frac{-12}{-20} = \frac{6}{10}$$

$$f(6) = \frac{12}{-20} = -0.6$$

$$f(5) = \frac{10}{16-25} = \frac{10}{-9}$$



Problem 12: (5pts) Write the range of function in the previous problem in interval notation.

all $y \in \mathbb{R}$ except $y = 0$ $\Rightarrow$ $\boxed{\text{range}(f(x)) = (-\infty, 0) \cup (0, \infty)}$