NAME ______________________________

MATH 114: FALL 2021 — **FINAL EXAM**

You may use the provided unit-circle and formula sheet. You are also allowed a page of notes.

Problem 1: (5pts) Suppose $\theta = 2\pi/9$ (in radians). Convert this angle to degrees.

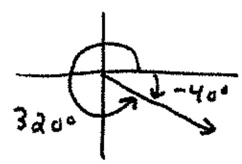
$$\theta = \left(\frac{2\pi}{9}\text{rad}\right)\left(\frac{180^\circ}{\pi\,\text{rad}}\right) = \boxed{40^\circ}$$

Problem 2: (5pts) Suppose $\theta = 150^o$. Convert this angle to radians.

$$\theta = (150\,\text{deg})\left(\frac{\pi\,\text{rad}}{180\,\text{deg}}\right) = \boxed{\frac{5\pi}{6}\text{rad.} \cong 2.618\,\text{rad}}$$

Problem 3: (5pts) Find the angle $0 < \alpha < 360^o$ which is coterminal with $\theta = -40^o$.

$$\alpha = -40^\circ + 360^\circ = \boxed{320^\circ}$$



Problem 4: (10pts) Consider a radius 10 cm circular arc which sweeps through a 150^o. Find:

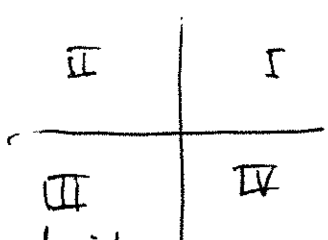
(a.) the arclength of the arc,

(b.) the area of the sector,

(a.) $$s = R\theta = (10\,\text{cm})\left(\frac{5\pi}{6}\text{rad}\right) = \boxed{\frac{25\pi}{3}\,\text{cm} \cong 26.18\,\text{cm}}$$

(b.) $$A = \frac{1}{2}R^2\theta = \frac{1}{2}(10\,\text{cm})^2\left(\frac{5\pi}{6}\right) = \boxed{\frac{125\pi}{3}\,\text{cm}^2 \cong 130.9\,\text{cm}^2}$$

Problem 5: (10pts) If $\cos t = -2/3$ and t is an angle in quadrant III then find the exact value of $\sin t$.

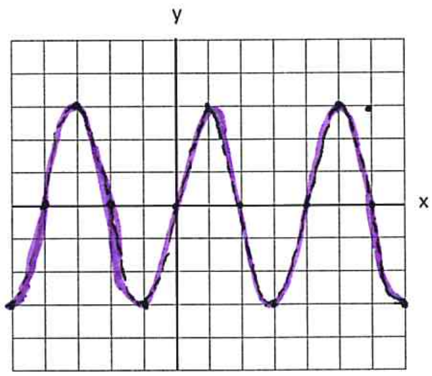


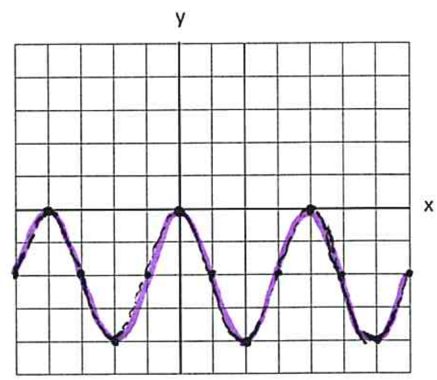
$\underbrace{\cos t, \sin t}_{\text{both negative}}$

$$\sin^2 t + \cos^2 t = 1$$
$$\sin t = -\sqrt{1-\cos^2 t}$$
$$= -\sqrt{1-\left(\frac{-2}{3}\right)^2}$$
$$= -\sqrt{1-\frac{4}{9}}$$
$$= \boxed{-\frac{\sqrt{5}}{3}}$$

$2\pi = \frac{\pi x}{2} \Rightarrow x = 4 \therefore \boxed{T = 4}$ is period of graphs below

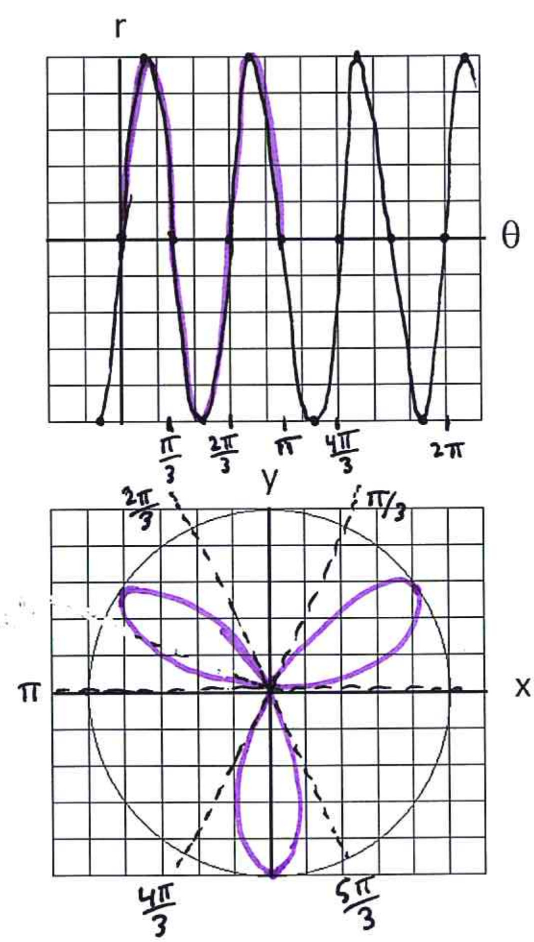
Problem 6: (10pts) Graph $y = 3\sin\left(\frac{\pi x}{2}\right)$ in the left grid and $y = 2\cos\left(\frac{\pi x}{2}\right) - 2$ in the right grid provided below. Also, answer the following question: what is the period of these sinusoidal functions ?



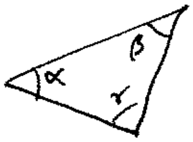


Problem 7: (10pts) Graph $r = 5\sin(3\theta)$ using the grids given below:

$3\theta = 2\pi \hookrightarrow \theta = \frac{2\pi}{3}$ is period of $\sin(3\theta)$



Problem 8: (10pts) If α, β and γ are angles in the same triangle, then prove that $\sin(\alpha+\beta) = \sin\gamma$.



$$\alpha + \beta + \gamma = \pi$$
$$\alpha + \beta = \pi - \gamma$$
$$\sin(\alpha+\beta) = \sin(\pi - \gamma)$$
$$= \underbrace{\cancel{\sin}\pi}_{0}\cos\gamma - \overbrace{\cancel{\cos}\pi}^{-1}\sin\gamma$$
$$= \underline{\sin\gamma}.//$$

Problem 9: (5pts) Simplify $\sin(-x)\sec(-x)\csc(-x)$.

$$\sin(-x)\sec(-x)\csc(-x) = \sec(-x)\frac{\sin(-x)}{\sin(-x)} = \sec(-x) = \frac{1}{\cos(-x)} = \boxed{\frac{1}{\cos(x)}}$$

Problem 10: (10pts) Simplify the expression below and leave your answer in terms of $\sin x$.

$$\frac{\sec x + \csc x}{1 + \tan x} = \left(\frac{\frac{1}{\cos x} + \frac{1}{\sin x}}{1 + \frac{\sin x}{\cos x}}\right)\left(\frac{\sin x \cos x}{\sin x \cos x}\right)$$
$$= \frac{\sin x + \cos x}{\sin x \cos x + \sin^2 x}$$
$$= \frac{\sin x + \cos x}{\sin x(\cos x + \sin x)}$$
$$= \boxed{\frac{1}{\sin x}}$$

Problem 11: (10pts) Use trigonometric identities to simplify the following expression:

$$\frac{\sin x \cos^2 x + \sin^3 x}{\csc x} + \cos^2 x = \frac{\sin(x)\left[\cos^2 x + \sin^2 x\right]}{\csc(x)} + \cos^2 x$$
$$= \frac{\sin(x)}{\frac{1}{\sin x}} + \cos^2 x$$
$$= \sin^2 x + \cos^2 x$$
$$= \boxed{1.}$$

Problem 12: (15pts) Use an appropriate identity to rewrite each of the following expressions:

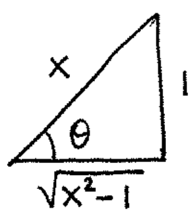
(A.) $\cos 10x \cos 8x + \sin 10x \sin 8x =$ $\cos(10x - 8x) = \boxed{\cos(2x)}$

(B.) $\cos 4x \sin 7x + \sin 4x \cos 7x =$ $\sin(4x + 7x) = \boxed{\sin(11x)}$

(C.) $\sin 2x + \sin 5x =$ $2\sin\left(\frac{2x+5x}{2}\right)\cos\left(\frac{2x-5x}{2}\right) = \boxed{2\sin\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)}$

Problem 13: (10pts) Find the exact value of $\tan\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$ in terms of x.

$\theta = \sin^{-1}\left(\frac{1}{x}\right) \hookrightarrow \sin\theta = \frac{1}{x}$



$\tan\theta = \frac{\text{OPP.}}{\text{ADJ.}} = \frac{1}{\sqrt{x^2-1}}$

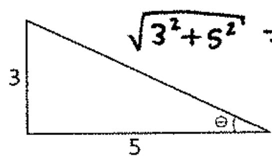
$$\boxed{\tan\left(\sin^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\sqrt{x^2-1}}}$$

Problem 14: (10pts) Write the range of each inverse function in interval notation on the blanks provided:

(A.) $range(\tan^{-1}) =$ $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(B.) $range(\cos^{-1}) =$ $[0, \pi]$

Problem 15: (10pts) Find the length of the hypotenuse and the angle θ in the triangle pictured below:



$\sqrt{3^2+5^2} = \boxed{\sqrt{34}}$, $\tan\theta = \frac{3}{5}$

$\theta = \tan^{-1}\left(\frac{3}{5}\right) \cong \boxed{30.96^\circ}$

Problem 16: (15pts) Find a solution in degrees or state no solution exists.

(A.) $\cos x = 1.001$

$|\cos x| < 1$ so $\cos x = 1.001 > 1$ impossible! $\boxed{\text{NO SOLUTION EXISTS}}$

(B.) $\sin x = 0.4$

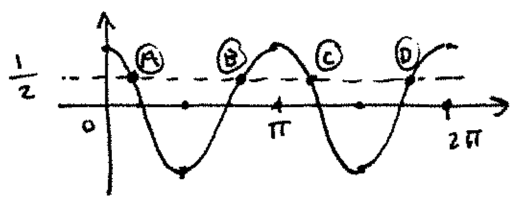
$$x = \sin^{-1}(0.4) \cong \boxed{23.58°}$$

(C.) $\tan x = 1$

$$x = \tan^{-1}(1) = \boxed{45°}$$

Problem 17: (10pts) Solve $\cos(2x) = \frac{1}{2}$ for $x \in [0, 2\pi]$.

$2x = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ $\therefore$ $\boxed{x = \frac{\pi}{6}}$ is a sol^{n}.



cos(2x) has period π. So we have 2 cycles on $[0, 2\pi]$

evidently there are four sol^{n}'s to find, we found Ⓐ already. to obtain Ⓒ add π since it is a full cycle ahead, Ⓒ: $\boxed{x = \pi + \frac{\pi}{6}}$
By symmetry, Ⓑ: $\pi - \frac{\pi}{6} = \boxed{\frac{5\pi}{6}}$ and thus Ⓓ $\boxed{x = \pi + \frac{5\pi}{6}}$

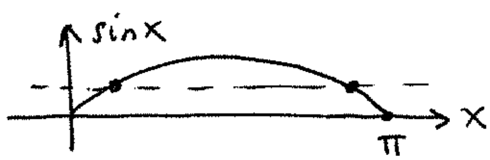
In summary, $\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$

Problem 18: (10pts) Find all solutions of $2\sin^2 x - 5\sin x + 2 = 0$ for $x \in [0, \pi]$.

$$(2\sin x - 1)(\sin x - 2) = 0$$

$$\Rightarrow 2\sin x - 1 = 0 \quad \text{or} \quad \sin x - 2 = 0$$

$$\Rightarrow \sin(x) = \frac{1}{2} \quad \text{or} \quad \underbrace{\sin(x) = 2}_{\text{NO SOLUTION.}}$$



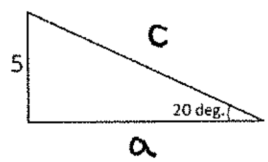
$$\boxed{x = \frac{\pi}{6}} \quad \text{or} \quad x = \pi - \frac{\pi}{6} = \boxed{\frac{5\pi}{6}}$$

Problem 19: (5pts) Find the polar form of the equation $x^2 + 3y + y^2 = 0$.

$$r^2\cos^2\theta + 3r\sin\theta + r^2\sin^2\theta = 0$$

aka $\boxed{r^2 + 3r\sin\theta = 0}$

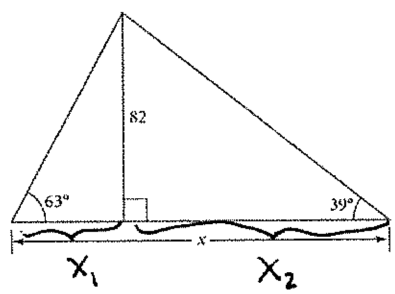
Problem 20: (10pts) Find the length of the hypotenuse and the adjacent side of the triangle below:



$$\sin(20^\circ) = \frac{5}{c} \quad \therefore \quad c = \frac{5}{\sin(20^\circ)} \cong \boxed{14.62}$$

$$\tan(20^\circ) = \frac{5}{a} \qquad a = \frac{5}{\tan(20^\circ)} \cong \boxed{13.74}$$

Problem 21: (10pts) Find x.

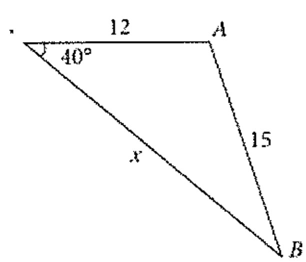


$$\tan 63^\circ = \frac{82}{x_1} \quad \therefore \quad x_1 = \frac{82}{\tan 63^\circ}$$

$$\tan 39^\circ = \frac{82}{x_2} \quad \therefore \quad x_2 = \frac{82}{\tan 39^\circ}$$

$$x = x_1 + x_2 = \frac{82}{\tan 63^\circ} + \frac{82}{\tan 39^\circ} \cong \boxed{143.04}$$

Problem 22: (10pts) Find x.



$$15^2 = 12^2 + x^2 - 2(12)x\cos(40^\circ)$$

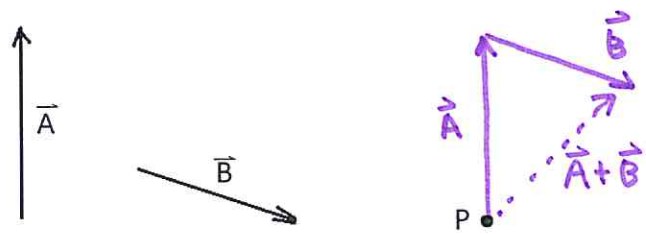
$$x^2 - 24\cos(40^\circ)x - 81 = 0$$

$$x = \frac{24\cos 40^\circ \pm \sqrt{(24\cos 40^\circ)^2 + 4(81)}}{2}$$

$\boxed{x \cong 22.06}$ or $\cancel{-3.672}$

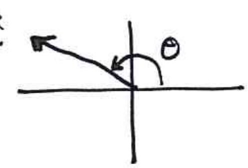
extraneous

Problem 23: (10pts) Vectors $\vec{A}$ and $\vec{B}$ are plotted below. Draw $\vec{A}+\vec{B}$ starting at the point P pictured below. Use the tip-to-tail method.



Problem 24: (20pts) Find the standard angle (in degrees) and magnitude of each of the following vectors:

(a.) $\vec{C} = \langle -2, 2\sqrt{3} \rangle$



$\tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) = -60^\circ$ add 180°

$\rightarrow \boxed{\theta = 120^\circ}$

$C = \sqrt{4+4(3)} = \sqrt{16} = \boxed{4}$

(b.) $\vec{D} = \langle 1, -1 \rangle$

$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = \boxed{-45^\circ}$

$D = \sqrt{1^2+1^2} = \boxed{\sqrt{2}}$

Problem 25: (10pts) If $\vec{A}$ has $A = 10$ and standard angle 30^o and $\vec{B}$ has $B = 5$ and standard angle 270^o then find the magnitude and standard angle of $\vec{A}+\vec{B}$.

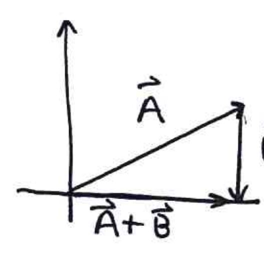
$\vec{A} = 10\langle \cos 30^\circ, \sin 30^\circ \rangle = \langle 8.660, 5 \rangle$

$\vec{B} = 5\langle \cos 270^\circ, \sin 270^\circ \rangle = 5\langle 0, -1 \rangle = \langle 0, -5 \rangle$

$\vec{A}+\vec{B} \cong \langle 8.660, 5 \rangle + \langle 0, -5 \rangle$

$\cong \langle 8.660, 5-5 \rangle$

$\cong \langle 8.660, 0 \rangle$



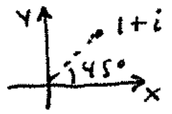
$||\vec{A}+\vec{B}|| \cong \sqrt{(8.66)^2+0^2} \cong \boxed{8.66 = 5\sqrt{3}}$

$\boxed{\theta = 0^\circ}$

Problem 26: (10pts) Write the following complex numbers in polar form ($z = re^{i\theta}$).

(a.) $z = 1 + i$,

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

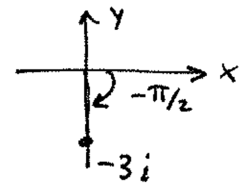


$$\therefore\ \theta = \frac{\pi}{4}$$

$$\boxed{z = \sqrt{2}\, e^{\frac{\pi i}{4}}}$$

(b.) $z = -3i$,

$$-i = e^{\frac{-\pi i}{2}} \quad \hookrightarrow \quad \boxed{z = 3e^{\frac{-\pi i}{2}}}$$



Problem 27: (10pts) Let $z = 2 + 3i$ and $w = -1 - 2i$. Find the Cartesian and polar forms of $(z + w)^8$.

$$z + w = (2 + 3i) + (-1 - 2i) = 1 + i = \sqrt{2}\, e^{\frac{\pi i}{4}}$$

$$(z + w)^8 = \left(\sqrt{2}\, e^{\frac{\pi i}{4}}\right)^8$$

$$= \left((\sqrt{2})^2\right)^4 e^{\frac{8\pi i}{4}}$$

$$= 2^4 e^{2\pi i} \leftarrow e^{2\pi i} = \underbrace{\cos 2\pi}_{1} + i \underbrace{\sin 2\pi}_{0} = 1$$

$$= \boxed{16}$$

Problem 28: (5pts) Use the formulas $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ to derive the identity
$\cos(2x)\sin(3x) = \dfrac{1}{2}\sin(x) + \dfrac{1}{2}\sin(5x)$.

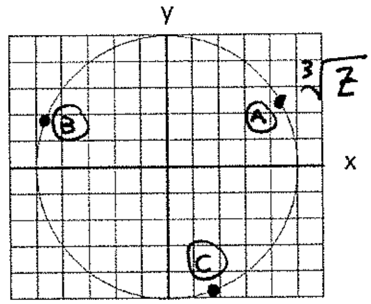
$$\cos(2x)\sin(3x) = \frac{1}{2}\left(e^{2ix} + e^{-2ix}\right)\frac{1}{2i}\left(e^{3ix} - e^{-3ix}\right)$$

$$= \frac{1}{4i}\left(e^{5ix} - e^{-5ix} + e^{ix} - e^{-ix}\right)$$

$$= \frac{1}{2}\left[\frac{1}{2i}\left(e^{5ix} - e^{-5ix}\right) + \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\right]$$

$$= \boxed{\frac{1}{2}\sin(5x) + \frac{1}{2}\sin(x)}$$

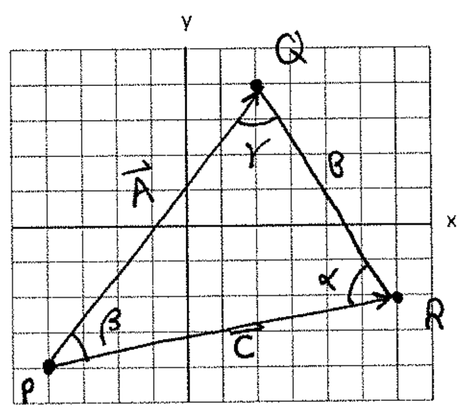
Problem 29: (10pt) Let $z = 125e^{3\pi i/6}$. Find the Cartesian form of $\sqrt[3]{z}$ and plot all three elements of $z^{1/3}$ in the plot below:



$$\sqrt[3]{z} = \sqrt[3]{125}\, e^{\frac{3\pi i}{3\cdot 6}}$$
$$= 5e^{\pi i/6}$$
$$= 5\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right)$$
$$= \boxed{5\left(\tfrac{\sqrt{3}}{2} + \tfrac{i}{2}\right)}$$

$$z^{\frac{1}{3}} = \left\{\sqrt[3]{z},\ \zeta_3\sqrt[3]{z},\ \zeta_3^2\sqrt[3]{z}\right\} = \left\{\underset{A}{\sqrt[3]{z}},\ \underset{B}{e^{\frac{2\pi i}{3}}\sqrt[3]{z}},\ \underset{C}{e^{\frac{4\pi i}{3}}\sqrt[3]{z}}\right\}$$

Problem 30: (20pts) Let $P = (-4, -4)$ and $Q = (2, 4)$ and $R = (6, -2)$. Find the perimeter, interior angles, and area of the triangle PQR. Is this triangle oblique ?



$$A = \overline{PQ} = \sqrt{6^2 + 8^2} = 10$$
$$B = \overline{QR} = \sqrt{4^2 + 6^2} = \sqrt{52}$$
$$C = \overline{PR} = \sqrt{10^2 + 2^2} = \sqrt{104}$$

perimeter $= A + B + C = 10 + \sqrt{52} + \sqrt{104}$

$\boxed{\text{perimeter} \cong 27.41}$

$$\vec{A} = Q - P = \langle 6, 8\rangle$$
$$\vec{C} = R - P = \langle 10, 2\rangle$$

$$\vec{A}\cdot\vec{C} = AC\cos\beta$$
$$60 + 16 = 10\sqrt{104}\cos\beta$$
$$\beta = \cos^{-1}\left(\frac{76}{10\sqrt{104}}\right) = \boxed{41.82^\circ}$$

Law of Sines

$$\frac{\sin\beta}{B} = \frac{\sin\alpha}{A} = \frac{\sin\gamma}{C}$$

$$\sin\alpha = \frac{A\sin\beta}{B} \Rightarrow$$
$$\therefore\ \alpha = \sin^{-1}\left(\frac{10\sin(41.82^\circ)}{\sqrt{52}}\right) \cong \boxed{67.62^\circ}$$

$$\sin\gamma = \frac{C\sin\beta}{B}$$
$$\therefore\ \gamma = \sin^{-1}\left(\frac{\sqrt{104}\sin(41.82^\circ)}{\sqrt{52}}\right) \cong \boxed{70.56^\circ}$$

$$\text{area} = \tfrac{1}{2}\|\vec{A}\times\vec{C}\| = \tfrac{1}{2}AC\sin(\beta)$$
$$= \tfrac{1}{2}(10)\sqrt{104}\sin(41.82^\circ) \cong \boxed{34.00} \leftarrow \text{area.}$$