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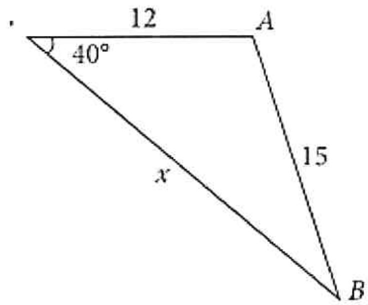
Math 114: Fall 2021 | **Test 3**

You may use the provided unit-circle and formula sheet. You are also allowed a 3x5 inch card of notes.

Problem 1: (5pts) Find x.



Law of Cosines

$(12\cos 40° \cong 9.193)$

$15^2 = 12^2 + x^2 - (12x\cos 40°)(2)$

$x^2 - 24x\cos 40° - 81 = 0$

$x = \dfrac{24\cos 40° \pm \sqrt{(24\cos 40°)^2 - 4(-81)}}{2}$

$\boxed{x \cong 22.05}$ or extraneous -3.67

Alternative Law of Sines Sol^n

$\dfrac{\sin 40}{15} = \dfrac{\sin B}{12}$

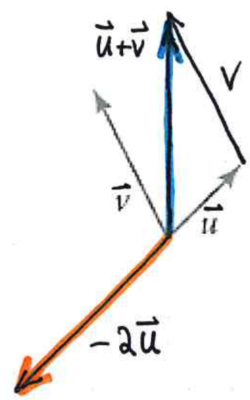
$B = \sin^{-1}\left(\dfrac{12\sin 40}{15}\right) \cong 30.95°$

Thus $A = 180° - 40° - 30.95° \cong 109.1°$

$\dfrac{\sin 40}{15} = \dfrac{\sin(109.1°)}{x} \Rightarrow x \cong 15\left(\dfrac{\sin 109.1°}{\sin 40°}\right)$

$\boxed{x \cong 22.05}$

Problem 2: Plot the vectors $\vec{u} + \vec{v}$ and $-2\vec{u}$ for $\vec{u}$ and $\vec{v}$ as given below:



Problem 3: Find the Cartesian form z_1 and z_2. Also, plot z_1 and z_2 as points in the graph.

(a.) $z_1 = 5e^{i\pi/3}$, $= 5\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 5\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \boxed{\frac{5}{2} + i\,\frac{5\sqrt{3}}{2}}$

$\cong 2.5 + 4.33i$

(b.) $|z_2| = 4$ and $\angle z_2 = 210^o$,

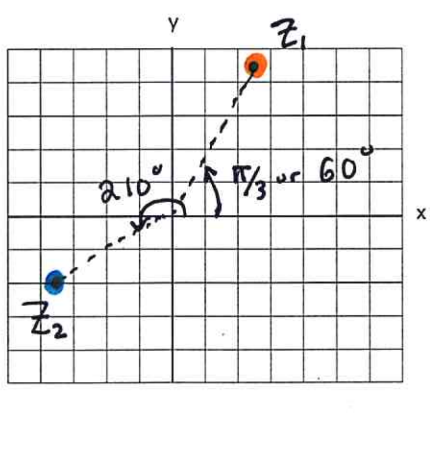
$z_2 = 4e^{\frac{7\pi i}{6}}$

$= 4\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$

$= 4\left(-\frac{\sqrt{3}}{2} + i\left(\frac{-1}{2}\right)\right)$

$= \boxed{-2\sqrt{3} - 2i}$

$= -3.46 - 2i$



Problem 4: If $\vec{A}$ has $A = 5$ and standard angle 45^o and $\vec{B}$ has $B = 5$ and standard angle 180^o then,

(a.) find the Cartesian forms of $\vec{A}$ and $\vec{B}$,

(b.) algebraically calculate $\vec{A} + \vec{B}$,

(c.) find the magnitude and standard angle of $\vec{A} + \vec{B}$,

(a.) $\vec{A} = \langle A\cos\theta, A\sin\theta \rangle = \langle 5\cos 45^\circ, 5\sin 45^\circ \rangle \cong \boxed{\langle 3.536, 3.536 \rangle}$

$\vec{B} = \langle B\cos\theta, B\sin\theta \rangle = \langle 5\cos 180^\circ, 5\sin 180^\circ \rangle = \boxed{\langle -5, 0 \rangle}$

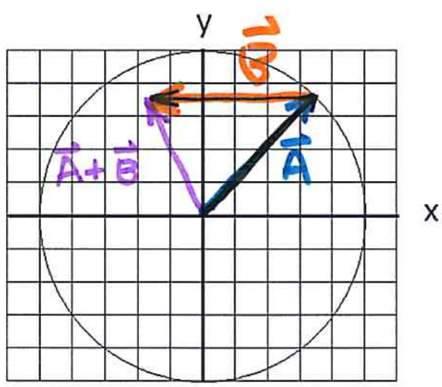
(b.) $\vec{A} + \vec{B} \cong \langle 3.536, 3.536 \rangle + \langle -5, 0 \rangle = \boxed{\langle -1.464, 3.536 \rangle}$

(c.) $\| \vec{A} + \vec{B} \| \cong \sqrt{(-1.464)^2 + (3.536)^2} \cong \boxed{3.83}$

$\tan^{-1}\left(\frac{3.536}{-1.464}\right) \cong -67.51^\circ$, $\vec{A} + \vec{B}$ is in quadrant II

Thus $\theta = 180^\circ - 67.51^\circ = \boxed{112.49^\circ}$

(d.) plot $\vec{A} + \vec{B}$ as it relates to $\vec{A}$ and $\vec{B}$ via the the tip-to-tail vector addition rule.



Problem 5: Write the following complex numbers in polar form.

(a.) $z = -2 - 2i$,

$180^\circ + 45^\circ = 225^\circ$

$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$z = -2 - 2i$

$|z| = \sqrt{4+4} = \sqrt{8}$

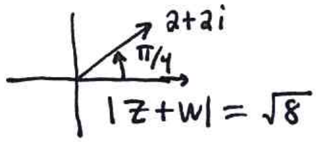
$\boxed{z = \sqrt{8}\, e^{\frac{5\pi i}{4}}}$

(b.) $z = 3i$,

$\boxed{z = 3e^{\frac{\pi i}{2}}}$ since $e^{\frac{\pi i}{2}} = \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + i\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} = i$

Problem 6: Let $z = 3 + i$ and $w = -1 + i$. Find the Cartesian and polar forms of $(z + w)^{10}$.

$$z + w = (3+i)+(-1+i) = 2 + 2i$$



$$\therefore\ z+w = \sqrt{8}\, e^{\frac{\pi i}{4}}$$

$$(z+w)^{10} = (\sqrt{8})^{10}\left(e^{\frac{\pi i}{4}}\right)^{10} = 32{,}768\, e^{\frac{10\pi i}{4}}$$

$$= 32{,}768\left(\cancelto{0}{\cos\left(\tfrac{10\pi}{4}\right)} + i\,\cancelto{1}{\sin\left(\tfrac{10\pi}{4}\right)}\right)$$

$$= \boxed{32768\, i} \quad \text{(CARTESIAN)}$$

$$= \boxed{32{,}768\, e^{\frac{\pi i}{2}}} \quad \text{(POLAR)}$$

Problem 7: Find the polar form of the equation $x^2 + 2x + y^2 = 0$.

$x = r\cos\theta\ ,\quad y = r\sin\theta$

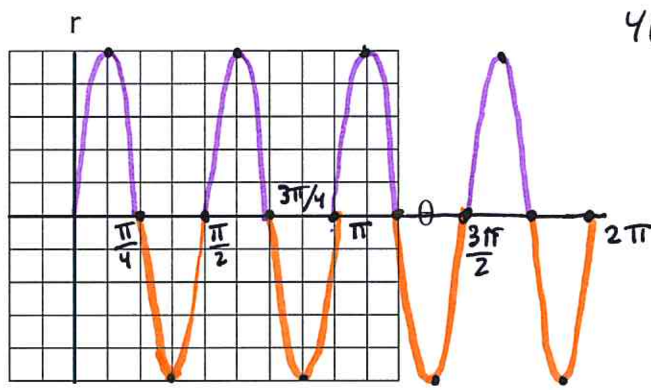
$$x^2 + 2x + y^2 = 0$$

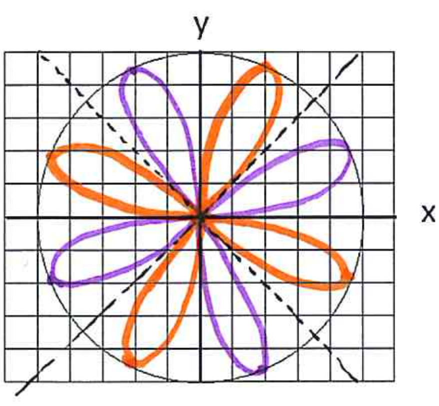
$$\Rightarrow\ r^2\cos^2\theta + 2r\cos\theta + r^2\sin^2\theta = 0$$

$$\Rightarrow\ \boxed{r^2 + 2r\cos\theta = 0}$$

Problem 8: Graph $r = 5\sin(4\theta)$ using the grids given below:

$4\theta = 2\pi \hookrightarrow \theta = \frac{\pi}{2}$ for full cycle.





Problem 9: (4pts) Find the standard angle (in degrees) and magnitude of each of the following vectors:

(a.) $\vec{C} = \langle -3, -4 \rangle$

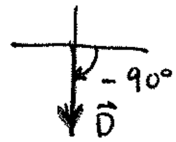
$C = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = \boxed{5}$

$\tan^{-1}\left(\frac{-4}{-3}\right) = \tan^{-1}\left(\frac{4}{3}\right) \cong 53.13°$

$\hookrightarrow \boxed{\theta \cong 233.13°}$

(b.) $\vec{D} = \langle 0, -10 \rangle$

$\boxed{D = 10, \quad \theta = -90°}$



Problem 10: Let $\vec{A} = \langle 1, 2, -2 \rangle$ and $\vec{B} = \langle 3, 0, 4 \rangle$.

(a.) find the magnitudes of $\vec{A}$ and $\vec{B}$
(b.) calculate $\vec{A} \bullet \vec{B}$
(c.) find the angle between $\vec{A}$ and $\vec{B}$
(d.) are the vectors parallel, perpendicular or neither ?

(a.) $A = \sqrt{1+4+4} = \sqrt{9} = \boxed{3}$

$B = \sqrt{3^2+0^2+4^2} = \sqrt{25} = \boxed{5}$

(b.) $\vec{A} \cdot \vec{B} = \langle 1, 2, -2 \rangle \cdot \langle 3, 0, 4 \rangle = 1 \cdot 3 + 2 \cdot 0 - 2 \cdot 4 = \boxed{-5}$

(c.) $\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{AB}\right) = \cos^{-1}\left(\frac{-5}{3 \cdot 5}\right) \cong \boxed{109.47°}$

(d.) $\vec{A}$ & $\vec{B}$ are neither $\parallel$ nor $\perp$.

Problem 11: Let $\vec{A} = \langle 1, 2, -2 \rangle$ and $\vec{B} = \langle 0, 1, 1 \rangle$. Calculate $\vec{A} \times \vec{B}$.

$$\vec{A} \times \vec{B} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

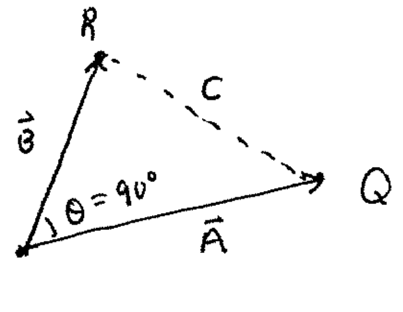
$$= \hat{x}(2+2) - \hat{y}(1-0) + \hat{z}(1-0)$$

$$= \boxed{\langle 4, -1, 1 \rangle}$$

<u>Check Answer</u>: $\langle 1,2,-2 \rangle \cdot \langle 4,-1,1 \rangle = 4-2-2 \stackrel{\checkmark}{=} 0$

$\langle 0,1,1 \rangle \cdot \langle 4,-1,1 \rangle = 0-1+1 \stackrel{\checkmark}{=} 0$

Problem 12: (2pts) Let $P = (0,0,0)$ and $Q = (1,3,4)$ and $R = (0,-4,3)$. Find the interior angles and the area of the triangle PQR. Is this triangle oblique ? *hint: use vectors*



$\vec{A} = \vec{PQ} = Q - P = \langle 1,3,4 \rangle$, $A = \sqrt{1+9+16}$

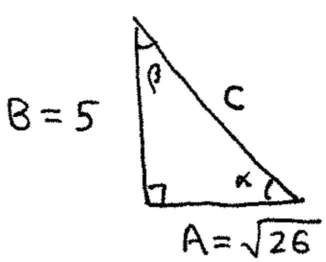
$\vec{B} = \vec{PR} = R - P = \langle 0,-4,3 \rangle$, $B = \sqrt{0+16+9}$

$\vec{A} \cdot \vec{B} = AB\cos\theta = 0$

$\theta = \cos^{-1}\left(\frac{0}{AB}\right) = \boxed{90°}$ $\therefore$ PQR is RIGHT TRIAN

$\boxed{\text{NOT OBLIQUE}}$

(OBLIQUE MEANS NOT RIGHT TRIANGE)



$$\text{area} = \frac{1}{2}(\text{BASE})(\text{HEIGHT})$$

$$= \frac{1}{2}\sqrt{26}\,(5)$$

$$= \boxed{\frac{5\sqrt{26}}{2} \cong 12.75}$$

$$\alpha = \tan^{-1}\left(\frac{5}{\sqrt{26}}\right) \cong \boxed{44.44°}$$

$$\beta = 180° - 90° - 44.44° \cong \boxed{45.56°}$$

(if this was not a right triangle, then we'd need to use the approach we did in Lecture 11/17/21)

Problem 13: Let $z = 625\exp(2\pi i/3)$. Calculate $\sqrt[4]{z}$ and all four complex numbers in $z^{1/4}$. Also, plot each answer in the complex plane provided below:

$$\sqrt[4]{z} = \sqrt[4]{625}\, e^{\frac{i\theta}{4}} \quad \text{where } \theta = \frac{2\pi}{3} \;\therefore\; \frac{\theta}{4} = \frac{2\pi}{12} = \frac{\pi}{6}$$

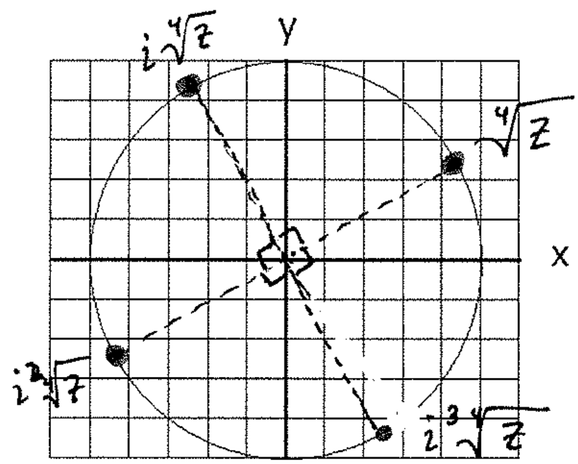
$$= \boxed{5\, e^{\pi i/6}}$$

$$\zeta_4 = \exp\left(\frac{2\pi i}{4}\right) = e^{\frac{\pi i}{2}} = i$$

$$z^{1/4} = \{\sqrt[4]{z},\ i\sqrt[4]{z},\ i^2\sqrt[4]{z},\ i^3\sqrt[4]{z}\}$$

$$= \{5e^{\pi i/6},\ 5\exp\left(\frac{2\pi i}{3}\right),\ 5\exp\left(\frac{7\pi i}{6}\right),\ 5\exp\left(\frac{5\pi i}{3}\right)\}$$

rotate by 90° to get to next member, multiplication by i is rotation by 90°.



Problem 14: Use the formulas $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ to derive the identity $\cos(2x)\sin(3x) = \dfrac{1}{2}\sin(x) + \dfrac{1}{2}\sin(5x)$.

$$\cos(2x)\sin(3x) = \frac{1}{2}\left(e^{2ix} + e^{-2ix}\right)\frac{1}{2i}\left(e^{3ix} - e^{-3ix}\right)$$

$$= \frac{1}{4i}\left(e^{5ix} - e^{-5ix} + e^{ix} - e^{-ix}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2i}\left(e^{5ix} - e^{-5ix}\right) + \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\right)$$

$$= \frac{1}{2}\left(\sin(5x) + \sin(x)\right)$$

$$= \frac{1}{2}\sin(x) + \frac{1}{2}\sin(5x). \;//$$