The text for this course is *Mathematical Analysis I* second edition by Beatriz Laferriere, Gerardo Laferriere and Nguyen Mau Nam.

Problem 1: Tell me something you learned from reading the article by Pete Clark.

Problem 2: Let A, B, C, D be sets. Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Problem 3: Let A, B, C, D be sets where $A \subseteq C$ and $B \subseteq D$. Prove $A \times B \subseteq C \times D$.

Problem 4: Let A, B, C be sets. Prove $A - (B \cup C) = (A - B) \cap (A - C)$.

Problem 5: Let $B, C \subseteq X$ where X is the universal set. Prove $\overline{B \cup C} = \overline{B} \cap \overline{C}$.
Hint: you may reference the result of the previous problem

Problem 6: Consider A, B, C finite sets. Let $\text{card}(A) = |A|$ denote the number of elements in A. Consider using an appropriate picture (Venn Diagram) to solve the following:

(a.) Explain why $|A \cup B| = |A| + |B| - |A \cap B|$.
(b.) Explain why $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

There are similar formulas for unions of more sets, however, Venn Diagrams are only easy to draw for up to 3 sets.

Problem 7: Let $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined by $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$. Prove or disprove that F is one-to-one. Prove or disprove that F is onto.

Problem 8: Exercise 1.2.2 from the text.

Problem 9: Exercise 1.2.3 from the text.

Problem 10: Exercise 1.2.4 from the text.

Problem 11: Exercise 1.2.7 (just part (d)) from the text.

Problem 12: Exercise 1.2.8 (just part (c)) from the text.

Problem 13: Let $A = [0, 2]$ and $B = \{1, 2, 3, 4\}$ define a relation on $\mathbb{R}$ by $R = A \times B \subseteq \mathbb{R} \times \mathbb{R}$.

(a.) find the domain of R
(b.) find the range of R
(c.) is R a function ?

Problem 14: Define $C_k \subseteq \mathbb{R}^2$ by $C_k = F^{-1}(\{k\})$ where $F(x, y) = x^2 + y^2$ and $k \in [0, \infty)$. For each $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, define $(x_1, y_1)R(x_2, y_2)$ if and only if there exists $k \in [0, \infty)$ such that $(x_1, y_1), (x_2, y_2) \in C_k$. Prove R is an equivalence relation on $\mathbb{R}^2$. Also, describe the equivalence classes of R and how they form a partition of $\mathbb{R}^2$.

Problem 15: Let $x, y \in \mathbb{Z}$ be R-related iff $y - x \in 3\mathbb{Z} = \{3k \mid k \in \mathbb{Z}\}$. Prove R is an equivalence relation on $\mathbb{Z}$. Also, describe the equivalence classes of R and how they partition $\mathbb{Z}$.

Problem 16: Suppose A and B each have n-elements. Prove a function $f : A \to B$ is injective iff f is surjective.

Problem 17: Suppose A and B are infinite sets with the same cardinality and suppose $f : A \to B$ is a function. If f is injective then is f surjective ? Likewise, if f is surjective then is f injective ? Discuss.

Problem 18: Explain how the cardinalities of the sets below are related. In particular, place the sets in order from smallest to greatest cardinality.

$$\mathbb{R}, (0, \infty), \mathbb{N}, [3, 7], \mathbb{Q}, \mathcal{P}(\mathbb{R}), \mathbb{Q} \times \mathbb{Q}, \{1, 2, 3, 4\}, \mathcal{P}(\{a, b\}), \emptyset$$

Problem 19: Find a bijection from $[0, 1]$ to $[4, 8]$.

Problem 20: Find a bijection from $(-\pi/2, \pi/2)^2$ to $\mathbb{R}^2$.

Problem 2: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof: Let $x \in A \cup (B \cap C)$ then $x \in A$ or $x \in B \cap C$.
If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ thus $x \in (A \cup B) \cap (A \cup C)$.
If $x \in B \cap C$ then $x \in A \cup B$ and $x \in A \cup C$ thus $x \in (A \cup B) \cap (A \cup C)$.

Hence $x \in (A \cup B) \cap (A \cup C)$ in all possible cases and we've shown $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Next, suppose $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Thus $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$. $*$
If $x \in A$ then $x \in A \cup (B \cap C)$. If $x \notin A$ then we must have $x \in B$ and $x \in C$ by $*$ and so $x \in B \cap C$.
Therefore, $x \in A \cup (B \cap C)$. So, in all cases, $x \in A \cup (B \cap C)$ and we've established $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
Hence, by double containment, $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$. //

Problem 3: Let A, B, C, D be sets with $A \subseteq C$ and $B \subseteq D$.
Suppose $x \in A \times B$ then $\exists a \in A, b \in B$ such that $x = (a, b)$. But $A \subseteq C$ hence $a \in C$ and $B \subseteq D$ hence $b \in D$ and so $(a, b) \in C \times D$. Thus $x \in C \times D$ and this proves $A \times B \subseteq C \times D$. //

Problem 4: $A - (B \cup C) = (A - B) \cap (A - C)$.

$$
\begin{aligned}
x \in A - (B \cup C) &\iff x \in A \text{ and } x \notin B \cup C \\
&\iff x \in A \text{ and } x \notin B \text{ and } x \notin C \\
&\iff (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\
&\iff x \in A - B \text{ and } x \in A - C \\
&\iff x \in (A - B) \cap (A - C)
\end{aligned}
$$

Thus $A - (B \cup C) = (A - B) \cap (A - C)$ since these sets have the same elements.

PROBLEM 5:

Let $B, C \subseteq X$ and define $\overline{B} = X - B$ and $\overline{C} = X - C$.

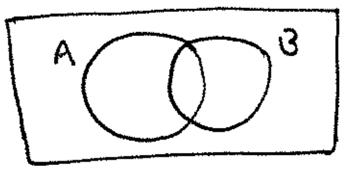
Then since $X - (B \cup C) = (X - B) \cap (X - C)$ by PROBLEM 4

we find $\overline{B \cup C} = \overline{B} \cap \overline{C}$. //

PROBLEM 6:

Let A, B, C be finite sets and denote $\text{card}(A) = \overline{\overline{A}} = |A|$.

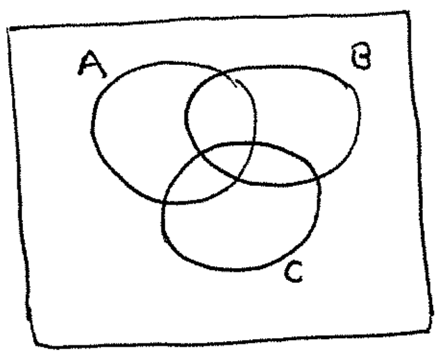
(a.)



$|A| + |B|$ double counts the elements in $A \cap B$ to count members of $A \cup B$, so subtract $|A \cap B|$ to correct,

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

(b.)



To count elements of $A \cup B \cup C$ we start with $|A| + |B| + |C|$ but that over counts overlaps so adjust by subtracting $|A \cap B|$, $|A \cap C|$ & $|B \cap C|$

But then, if we consider $x \in A \cap B \cap C$ notice it is counted by $|A|, |B|, |C|$ and $|A \cap B|$, $|A \cap C|$, $|A \cap C|$ so the formula needs to add $|A \cap B \cap C|$ to be correct,

$$\boxed{|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|}$$

Perhaps more convincing to give table of counting,

DISJOINT CASES	$\|A\|$	$\|B\|$	$\|C\|$	$-\|A \cap B\|$	$-\|A \cap C\|$	$-\|B \cap C\|$	$\|A \cap B \cap C\|$
$A - (B \cup C)$	1	0	0	0	0	0	0
$B - (A \cup C)$	0	1	0	0	0	0	0
$C - (A \cup B)$	0	0	1	0	0	0	0
$(A \cap B) - C = A \cap B - (A \cap B \cap C)$	1	1	0	-1	0	0	0
$(A \cap C) - B$	1	0	1	0	-1	0	0
$(B \cap C) - A$	0	1	1	0	0	-1	0
$A \cap B \cap C$	1	1	1	-1	-1	-1	1

PROBLEM 7:

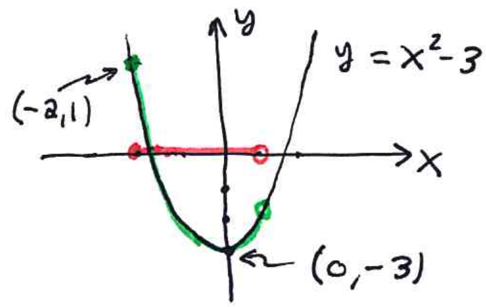
Let $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined by $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

Notice $F\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = F\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 1$ yet $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ thus F is not injective. In contrast, if $x \in \mathbb{R}$ then notice $F\left(\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}\right) = x\cdot 1 - 0\cdot 0 = x$ thus F is onto.

PROBLEM 8: (Ex. 1.2.2 from text)

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 - 3$ and let $A = [-2, 1)$ and $B = (-1, 6)$. Find $f(A)$ and $f^{-1}(B)$

$$\begin{aligned} f(A) &= \{ f(x) \mid x \in A \} \\ &= \{ x^2 - 3 \mid x \in [-2, 1) \} \\ &= \{ x^2 - 3 \mid -2 \le x < 1 \} \\ &= [-3, 1] \end{aligned}$$



I'll be content with the above proof by picture.

$$\begin{aligned} f^{-1}(B) &= \{ x \in \mathbb{R} \mid f(x) \in B \} \\ &= \{ x \in \mathbb{R} \mid x^2 - 3 \in (-1, 6) \} \end{aligned}$$

We need to solve $-1 < x^2 - 3 < 6$

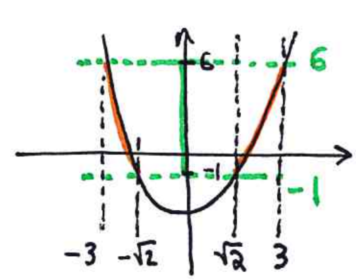
① $-1 < x^2 - 3 \Rightarrow 2 < x^2 \Rightarrow |x|^2 > 2$, $|x| > \sqrt{2} \Rightarrow x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

② $x^2 - 3 < 6 \Rightarrow x^2 < 9 \Rightarrow |x| < 3 \Rightarrow x \in (-3, 3)$

We need both ① and ② thus form the intersection,

$$(-3, 3) \cap [(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)] = (-3, -\sqrt{2}) \cup (\sqrt{2}, 3)$$

$$\Rightarrow \boxed{f^{-1}(B) = (-3, -\sqrt{2}) \cup (\sqrt{2}, 3)}$$



PROBLEM 9: Exercise 1.2.3, Prove the following fncts are bijective

(a.) $f : (-\infty, 3] \longrightarrow [-2, \infty)$ given by $f(x) = |x-3| - 2$

(b.) $g : (1,2) \longrightarrow (3,\infty)$ given by $g(x) = \frac{3}{x-1}$

(a.) Suppose $a, b \in (-\infty, 3]$ and $f(a) = f(b)$. Then $|a-3| - 2 = |b-3| - 2$ implies $|a-3| = |b-3|$ hence $a - 3 = \pm(b-3)$. If $(+)$ then $a-3 = b-3$ hence $a = b$. If $(-)$ then $a-3 = (-(b-3)) = -b+3$ which gives $a+b = 6$ but $a, b \leq 3$ so we find $a = b = 3$ in this case. Thus f is injective.

Let $y \in [-2, \infty)$ and note $-2 \leq y \Rightarrow -y \leq 2 \Rightarrow 1-y \leq 3$ thus $x = 1-y \in (-\infty, 3]$. Note $f(x) = |$

$$\begin{aligned} f(x) = f(1-y) &= |1-y-3| - 2 \\ &= |-y-2| - 2 \\ &= |y+2| - 2 \\ &= y + 2 - 2 \\ &= y. \end{aligned}$$

$-y \leq 2 \Rightarrow 0 \leq y+2.$ $\therefore |y+2| = y+2.$

Thus f is surjective and it follows f is bijection.

(b.) Suppose $a, b \in (1,2)$ and $g(a) = g(b)$ then $\frac{3}{a-1} = \frac{3}{b-1}$ hence $3(b-1) = 3(a-1) \Rightarrow 3b = 3a \Rightarrow a = b$. Thus g is 1-1.

Let $y \in (3, \infty)$

Then $y > 3 \Rightarrow 1 > 3/y > 0 \Rightarrow 2 > 1 + 3/y > 1$ thus $1 + 3/y \in (1,2) = \text{domain}(g)$. Furthermore,

$$g(1 + 3/y) = \frac{3}{1 + 3/y - 1} = \frac{3}{3/y} = y.$$

Therefore g is onto. Since g is 1-1 and onto it follows g is bijection.

PROBLEM 10: Ex. 1.2.4

Prove that if $f: X \to Y$ is injective, then the following hold:

(a.) $f(A \cap B) = f(A) \cap f(B)$ for $A, B \subseteq X$

(b.) $f(A - B) = f(A) - f(B)$ for $A, B \subseteq X$

(a.) Suppose f is 1-1. Let $y \in f(A \cap B)$ then $\exists\, x \in A \cap B$ such that $f(x) = y$. Then $x \in A$ and $x \in B$ hence $y \in f(A)$ and $y \in f(B)$ which proves $y \in f(A) \cap f(B)$. Therefore, $\underline{f(A \cap B) \subseteq f(A) \cap f(B)}$. (true for noninjective functions just the same)

Next, suppose $y \in f(A) \cap f(B)$ then $y \in f(A)$ and $y \in f(B)$ hence $\exists\, a \in A$ and $b \in B$ for which $y = f(a)$ and $y = f(b)$. Thus $f(a) = f(b)$ and since f is injective we find $a = b \in A \cap B$. This shows $y \in f(A \cap B)$. Consequently, $\underline{f(A) \cap f(B) \subseteq f(A \cap B)}$.

Therefore, $f(A \cap B) = f(A) \cap f(B)$ by double-containment.

(b.) Suppose f is 1-1. Let $y \in f(A - B)$ then $\exists\, x \in A - B$ such that $y = f(x)$. But, $x \in A - B$ gives $x \in A$ and $x \notin B$ hence $y \in f(A)$ and $y \notin f(B)$.* Thus $y \in f(A) - f(B)$. Therefore $f(A - B) \subseteq f(A) - f(B)$.

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Let $y \in f(A) - f(B)$ then $y \in f(A)$ and $y \notin f(B)$ hence $\exists\, a \in A$ for which $y = f(a)$ and $\nexists\, b \in B$ for which $f(b) = y$. It follows $a \in A - B$ thus $y \in f(A - B)$. Therefore $f(A) - f(B) \subseteq f(A - B) \Rightarrow f(A) - f(B) = f(A - B)$.

where is the gap in this argument?

Notice I've not used injectivity yet.

Here's the fix

$\boxed{\begin{array}{l} * \text{ why is } y \notin f(B)\,? \\ \text{Well, suppose } y \in f(B) \text{ then } y = f(b) \text{ for some } b \in B \\ \text{and as } y = f(x) = f(b) \Rightarrow x = b \text{ but } x \in A - B \\ \text{thus } x \notin B \text{ yet } b \in B \text{ which is a } \rightarrow\leftarrow \end{array}}$

PROBLEM 11: Exercise 1.2.7 prove Thm 1.2.5 part d.

Let $f : X \to Y$ be a function and $\{B_\beta\}_{\beta \in J}$ an indexed family of subsets of Y. Consider,

$$y \in f^{-1}\left(\bigcap_{\beta \in J} B_\beta\right) \iff \exists x \in \bigcap_{\beta \in J} B_\beta \text{ such that } f(x) = y$$

$$\iff \exists x \in B_\beta \ \forall \beta \in J \text{ such that } f(x) = y$$

$$\iff y \in f^{-1}(B_\beta) \ \forall \beta \in J$$

$$\iff y \in \bigcap_{\beta \in J} f^{-1}(B_\beta)$$

Consequently, $f^{-1}\left(\bigcap_{\beta \in J} B_\beta\right) = \bigcap_{\beta \in J} f^{-1}(B_\beta)$. //

PROBLEM 12: Ex. 1.2.8 prove part c of Thm 1.2.6

If f and g are surjective then $g \circ f$ is surjective

Proof: recall $f : X \to Y$ and $g : Y \to Z$ are functions.

Suppose f and g are onto. Notice $g \circ f : X \xrightarrow{f} Y \xrightarrow{g} Z$ is a function. Let $z \in Z$ then since g is onto $\exists y \in Y$ such that $g(y) = z$. But, f is onto hence $\exists x \in X$ such that $f(x) = y$. Observe

$$(g \circ f)(x) = g(f(x)) = g(y) = z$$

Thus $g \circ f$ is onto. //

PROBLEM 13

$A = [0, 2]$ and $B = \{1, 2, 3, 4\}$
and define a relation on $\mathbb{R}$ by $R = A \times B \subseteq \mathbb{R} \times \mathbb{R}$

(a.) domain $(R) = A = [0, 2]$

(b.) range $(R) = B = \{1,2,3,4\}$

(c.) Let us consider $(0,1), (0,2), (0,3), (0,4) \in R$

Thus R is not a function since the input of 0 gives four outputs.

PROBLEM 14

$C_k \subseteq \mathbb{R}^2$ by $C_k = F^{-1}(\{k\})$ where $F(x,y) = x^2 + y^2$
and $k \in [0, \infty)$. We define $(x_1, y_1) R (x_2, y_2)$ iff $\exists k \in [0, \infty)$
such that $(x_1, y_1), (x_2, y_2) \in C_k$.

① Let $(x,y) \in \mathbb{R}^2$ then let $k = x^2 + y^2$ and note $x^2 + y^2 \geq 0$ hence $k \geq 0$.
Since $(x,y) \in C_k$ we find $(x,y) R (x,y)$ which shows R is reflexive.

② Suppose $(x_1, y_1) R (x_2, y_2)$ then $(x_1, y_1), (x_2, y_2) \in C_k$ for some $k \geq 0$
Then $(x_2, y_2), (x_1, y_1) \in C_k$ shows $(x_2, y_2) R (x_1, y_1)$ hence
the relation R is symmetric.

③ Suppose $(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3)$ then
$(x_1, y_1), (x_2, y_2) \in C_{k_1}$ and $(x_2, y_2), (x_3, y_3) \in C_{k_2}$
thus $x_1^2 + y_1^2 = x_2^2 + y_2^2 = k_1$ and $x_2^2 + y_2^2 = x_3^2 + y_3^2 = k_2$
Thus $k_1 = k_2$ and we find $(x_1, y_1), (x_3, y_3) \in C_{k_1}$ $(k_1 = k_2)$
Hence $(x_1, y_1) R (x_3, y_3)$. Thus R is transitive

Therefore R is an equivalence relation.

Equivalence classes are circles and the origin.

$$[(x,y)] = \underbrace{\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = k = R^2\}}_{\text{concentric circles about the origin}}$$

and $[(0,0)] = \{(0,0)\}$.