PROBLEM 15:

Let $x, y \in \mathbb{Z}$ be R-related iff $y - x \in 3\mathbb{Z}$.

Let $x \in \mathbb{Z}$ and note $x - x = 0 = 3(0) \in 3\mathbb{Z}$ thus xRx and we've shown R is reflexive.

Suppose xRy then $y - x = 3j$ for some $j \in \mathbb{Z}$ thus $x - y = 3(-j) \in 3\mathbb{Z}$ hence yRx. Thus R is symmetric.

Suppose xRy and yRz then $\exists j, k \in \mathbb{Z}$ for which $y - x = 3j$ and $z - y = 3k$. Consider that,

$$\begin{aligned} z - x &= (y + 3k) + (3j - y) \quad (z = y + 3k \ \& \ -x = 3j - y) \\ &= 3(k + j) \end{aligned}$$

Then we see $z - x = 3(k+j) \in 3\mathbb{Z}$ and xRz. Therefore R is transitive and we've shown R is an equivalence relation.

$$\begin{aligned} [x] &= \{ y \in \mathbb{Z} \mid y - x \in 3\mathbb{Z} \} \\ &= \{ y \in \mathbb{Z} \mid \exists j \in \mathbb{Z} \text{ s.t. } y - x = 3j \} \\ &= \{ x + 3j \mid j \in \mathbb{Z} \} \\ &= x + 3\mathbb{Z} \end{aligned}$$

Note $[0] = 3\mathbb{Z}$, $[1] = 1 + 3\mathbb{Z}$ and $[2] = 2 + 3\mathbb{Z}$.

$$\begin{aligned} \mathbb{Z} &= 3\mathbb{Z} \cup (1 + 3\mathbb{Z}) \cup (2 + 3\mathbb{Z}) \\ &= \{\dots, -3, 0, 3, 6, \dots\} \cup \{\dots, -2, 1, 4, 7, \dots\} \cup \{\dots, -1, 2, 5, 8, \dots\} \end{aligned}$$

PROBLEM 16:

Let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$.

Suppose $f : A \to B$ is a function.

Further suppose f is injective and consider the image

$f(A) = \{f(a) \mid a \in A\} = \{f(a_1), f(a_2), \dots, f(a_n)\} \subseteq B$

Suppose $f(a_j) = f(a_k)$ then $a_j = a_k$ as f is 1-1.

Thus the elements $f(a_1), f(a_2), \dots, f(a_n)$ are distinct.

We find $|f(A)| = n$ thus $f(A) = B$.

(we cannot have $f(A) \subseteq B$ and $|f(A)| = |B|$ unless $f(A) = B$)

Thus f 1-1 $\Rightarrow$ f onto.

Conversely suppose f is onto. Then $f(A) = B$ and $\{f(a_1), f(a_2), \dots, f(a_n)\} = \{b_1, b_2, \dots, b_n\}$. Suppose $f(a_j) = f(a_k)$ and $a_j \neq a_k$ then the set $\{f(a_1), \dots, f(a_n)\}$ has at most $(n-1)$-distinct elements so $f(A) \neq B$ which $\rightarrow\leftarrow$ $f(A) = B$. Thus $f(a_j) = f(a_k)$ implies $a_j = a_k$ and so f is injective.

Therefore f onto $\Rightarrow$ f 1-1.

PROBLEM 17: Let A, B be infinite sets.

Suppose $|A| = |B|$ and $f: A \to B$ is a function.

Consider $A = B = \mathbb{Z}$.

1.) $f(x) = 2x$ defines $f: \mathbb{Z} \to \mathbb{Z}$ and

$f(a) = f(b) \Rightarrow 2a = 2b \Rightarrow a = b$ thus f - injective.

However, $3 \notin f(\mathbb{Z})$ since $2x = 3 \Rightarrow x = \frac{3}{2} \notin \mathbb{Z}$.

Therefore f is not a surjection.

2.) Let $f(x) = \begin{cases} j & : x = 2j \text{ for some } j \in \mathbb{Z} \\ j & : x = 2j+1 \text{ for some } j \in \mathbb{Z} \end{cases}$

Then we may argue f is onto $\mathbb{Z}$ yet
$f(2) = f(3)$ as $2 = 2(1)$ and $3 = 2(1) + 1$
So f is not injective.

$(f(0) = f(1) = 0,\ f(2) = f(3) = 1,\ f(4) = f(5) = 2 \text{ etc...})$

In short, PROBLEM 16 illustrates that finite sets are special.
In contrast, injective and surjective are not interchangeable
for $f: A \to B$ with $|A| = |B| \neq n$.

PROBLEM 18

$\mathbb{R}$, $(0, \infty)$, $\mathbb{N}$, $[3,7]$, $\mathbb{Q}$, $\mathcal{P}(\mathbb{R})$, $\mathbb{Q} \times \mathbb{Q}$, $\{1,2,3,4\}$, $\mathcal{P}(\{a,b\})$, $\emptyset$

Notice,

$0 = |\emptyset| < 4 = |\{1,2,3,4\}| = |\mathcal{P}(\{a,b\})| < \aleph_0 = |\mathbb{N}|$

$= |\mathbb{Q}| = |\mathbb{Q} \times \mathbb{Q}| < \aleph_1 = |\mathbb{R}| = |(0,\infty)| = |[3,7]| < |\mathcal{P}(\mathbb{R})|$

PROBLEM 19:

Find bijection from $[0,1]$ to $[4,8]$

$f(0) = 4$ & $f(1) = 8$

Let $f(x) = mx + b$

$f(0) = m(0) + b = 4 \therefore \underline{b = 4}$.

$f(1) = m + 4 = 8 \therefore \underline{m = 4}$

Hence $f(x) = 4x + 4$.

Notice $f(a) = f(b) \Rightarrow 4a + 4 = 4b + 4 \Rightarrow a = b \therefore f$ 1-1.

Likewise, if $y \in [4,8]$ then solve $y = 4x + 4$ for $x = \frac{y-4}{4}$

and note $4 \leq y \leq 8 \Rightarrow 1 \leq \frac{y}{4} \leq 2 \Rightarrow 0 \leq \frac{y}{4} - 1 \leq 1$

hence $f(\frac{y}{4} - 1) = 4(y/4 - 1) + 4 = y$ for $\frac{y}{4} - 1 \in [0,1]$.

Thus f is bijection.

PROBLEM 20:

Find bijection from $(-\pi/2, \pi/2)^2 \longrightarrow \mathbb{R}^2$.

Let $F(\theta, \beta) = (\tan\theta, \tan\beta)$

Then $F(\theta,\beta) = F(\theta', \beta') \Rightarrow (\tan\theta, \tan\beta) = (\tan\theta', \tan\beta')$

So $\tan\theta = \tan\theta'$ and $\tan\beta = \tan\beta'$ and for

$(\theta, \beta), (\theta', \beta') \in (-\pi/2, \pi/2)^2$ we have $-\frac{\pi}{2} < \theta, \theta', \beta, \beta' < \frac{\pi}{2}$

and thus $\theta = \theta'$ and $\beta = \beta' \Rightarrow (\theta, \beta) = (\theta', \beta') \therefore F$ 1-1.

Let $(x,y) \in \mathbb{R}^2$ and notice $(\tan^{-1}(x), \tan^{-1}(y)) \in (-\frac{\pi}{2}, \frac{\pi}{2})^2$

has $F(\tan^{-1}(x), \tan^{-1}(y)) = (\tan(\tan^{-1}(x)), \tan(\tan^{-1}(y))) = (x,y)$.

Thus F is onto. Consequently f is bijection. //